Chapter 1

Introduction

Literature: Görtz-Wedhorn: Algebraic Geometry I and II

The goal of this lecture is a brief introduction to the theory of group schemes.

Definition 1.1 (Group object)**.** Let $\mathcal{C}$ be a category with finite products. A *group object in* $\mathcal{C}$ is the data (G, m, e, i) where

- G is an object of $\mathcal{C}$
- $m\colon G \times G \to G$ is the multiplication map
- $e\colon 1 \to G$ is the unit
- $i\colon G \to G$ is the inversion map

such that the following diagrams commute

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \mathrm{id}} & G \times G \\ \big\downarrow{\scriptstyle \mathrm{id} \times m} & & \big\downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G \end{array}, \qquad \begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ {\scriptstyle \mathrm{id} \times e}\big\uparrow & \swarrow & \\ G \times 1 & & \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{\mathrm{id} \times i} & G \times G \\ \big\downarrow & & \big\downarrow{\scriptstyle m} \\ 1 & \xrightarrow{\quad e \quad} & G \end{array}.$$

G is called *commutative*, if additionally the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\mathrm{swap}} & G \times G \\ \big\downarrow{\scriptstyle m} & \swarrow{\scriptstyle m} & \\ G & & \end{array}$$

commutes.

A *morphism of group objects* $(G, m, e, i) \to (G', m', e', i')$ is a morphism $f\colon G \to G'$ in $\mathcal{C}$ such that the diagrams

$$\begin{array}{ccc} G \times G & \xrightarrow{f \times f} & G' \times G' \\ \big\downarrow{\scriptstyle m} & & \big\downarrow{\scriptstyle m'} \\ G & \xrightarrow{\quad f \quad} & G' \end{array}, \qquad \begin{array}{ccc} G & \xrightarrow{f} & G' \\ {\scriptstyle e}\big\uparrow & \nearrow{\scriptstyle e'} & \\ 1 & & \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \xrightarrow{f} & G' \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i'} \\ G & \xrightarrow{f} & G' \end{array}.$$

This defines the category $\mathrm{Grp}(\mathcal{C})$ of group objects in $\mathcal{C}$.

Example 1.2.
- $\mathrm{Grp}(\mathrm{Set}) \simeq \mathrm{Grp}$
- $\mathrm{Grp}(\mathrm{Grp}) \simeq \mathrm{Ab}$
- $\mathrm{Grp}(\mathrm{Ab}) \simeq ?$

- Grp(Top) $\simeq$ topological Groups
- Grp(Mfd_∞) $\simeq$ Lie Groups

Definition 1.3 (group scheme)**.** Let S be a scheme. An *S-group scheme* or an *S-group* is a group object in the category of schemes over S.

Remark 1.4. Let S be a scheme. The structure of a group scheme over S on a S-scheme G is equivalent to a factorisation of the functor of points

$$\begin{array}{ccc} \mathrm{Sch}_S & \longrightarrow & \mathrm{Set} \\ \downarrow & \nearrow & \\ \mathrm{Grp} & & \end{array}$$

via the forgetful functor from groups to sets.

Example 1.5. Let S be a scheme.

(i) Let Γ be a group. Then $G = \Gamma_S$ where $G(T) := \{$ locally constant maps $T \to \Gamma$ $\}$

(ii) (additive group) $\mathbb{G}_{a,S}$ where $\mathbb{G}_{a,S}(T) = \mathcal{O}_T(T)$. We have $\mathbb{G}_{a,S} \simeq \mathbb{A}^1_S$.

(iii) (multiplicative group) $\mathbb{G}_{m,S}$ where $\mathbb{G}_{m,S}(T) := \mathcal{O}_T(T)^\times$.

(iv) (roots of unity) $\mu_{n,S}$ $(n \geq 1)$ where $\mu_{n,S}(T) = \{x \in \mathcal{O}_T(T)^\times \mid x^n = 1\}$.

(v) $S = \mathrm{Spec}(R)$. $\mathrm{GL}_{n,R} = \mathrm{Spec}(A)$ where $A = R[T_{ij} \mid 1 \leq i,j \leq n][\det^{-1}]$ where $\det = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) T_{1\sigma(1)} \cdots T_{n\sigma(n)}$. We obtain $\mathrm{GL}_{n,S}$ by base changing $\mathrm{GL}_{n,\mathbb{Z}}$.

Lemma 1.6. *Let G be a S-group. Then $G \to S$ is separated if and only if $S \xrightarrow{e} G$ is a closed immersion.*

Definition 1.7. Let R be a ring. A *(commutative) Hopf-Algebra* over R is a group object in $\mathrm{Alg}_R^{\mathrm{op}}$, where $\mathrm{Alg}_R = \mathrm{CRing}_R$.

Remark 1.8. For a R-Hopf-Algebra A, we denote the canonical maps by

- $\mu\colon A \to A \otimes_R A$ (Comultiplication)
- $\varepsilon\colon A \to R$ (Counit)
- $\iota\colon A \to A$ (Antipode)

A Hopf-Algebra is called *cocommutative*, if the associated group object in $\mathrm{Alg}_R^{\mathrm{op}}$ kommutativ ist.

Remark 1.9. For a ring R, by construction we have an equivalence of categories between the category of affine R-group schemes and the opposite category of R-Hopf-Algebras.

Example 1.10. The additive group $\mathbb{G}_{a,R} = \mathrm{Spec}(R[t])$ has

- comultiplication $\mu\colon R[t] \to R[t] \otimes_R R[t], t \mapsto t \otimes 1 - 1 \otimes t$.
- counit $\varepsilon\colon R[t] \to R, t \mapsto 0$
- antipode $\iota\colon R[t] \to R[t], t \mapsto -t$

Proof. For any R-algebra A we have $\mathbb{G}_{a,R}(A) = A$ and the diagram

$$\begin{array}{ccc} \mathbb{G}_{a,R}(A) \times \mathbb{G}_{a,R}(A) & \xrightarrow{m} & \mathbb{G}_{a,R}(A) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}_R(R[s_1,s_2],A) & \xrightarrow{\mu^*} & \mathrm{Hom}_R(R[t],A) \end{array}.$$

□

Definition 1.11. Let G be a S-group. A *subgroupscheme* of G is a subscheme $H \subseteq G$ such that

1) for all $T \in \mathrm{Sch}_S$, we have $H(T) \subseteq G(T)$ a subgroup,
2) We have commutative diagrams

$$\begin{array}{ccccc} H \times_S H & \longrightarrow & G \times_S G & \xrightarrow{m} & G \\ \downarrow & & & \nearrow & \\ H & & & & \end{array} \quad \text{and} \quad \begin{array}{ccc} S & \xrightarrow{e} & G \\ \downarrow & \nearrow & \\ H & & \end{array}$$

A subgroup scheme $H \subseteq G$ is *normal* if $H(T)$ is a normal subgroup of $G(T)$ for all $T \in \mathrm{Sch}_S$.

For a morphism $f \colon G \to G'$ of S-groups and a subgroup $H' \subseteq G'$, let $f^{-1}(H')$ be $G \times'_G H$. For $H' = 1 \xrightarrow{e} G'$, we obtain the *kernel of* f and the cartesian square

$$\begin{array}{ccc} \mathrm{Ker}(f) & \longrightarrow & G \\ \downarrow & & \downarrow f \\ S & \xrightarrow{e} & G' \end{array} .$$

Remark 1.12. The kernel of a homomorphism f of S-groups is for any S-scheme T given by

$$\mathrm{Ker}(f)(T) = \ker\left(f(T)\right).$$

In particular, the $\mathrm{Ker}(f)$ is normal.

Definition 1.13. Let G be a S-group, T a S-scheme and $g \in G(T) = \mathrm{Hom}_S(T, G)$. Define the *lefttranslation by* g as

$$\begin{array}{ccc} G_T & \xleftarrow{=} & T \times_T G_T \\ \downarrow t_g & & \downarrow g \times \mathrm{id} \\ G_T & \xleftarrow[m]{} & G_T \times_T G_T \end{array} .$$

Remark 1.14. In the situation of 1.13, for every $T' \xrightarrow{f} T$, the map

$$t_g(T') \colon G_T(T') = G(T') \longrightarrow G(T') = G_T(T')$$

is the lefttranslation by the element $f^*(g) \in G(T')$.

Remark 1.15. Consider

$$\begin{array}{ccc} G \times_S G & \xrightarrow{(g,h) \mapsto (gh,h)} & G \times_S G \\ \downarrow m & \swarrow \mathrm{pr}_1 & \\ G & & \end{array} .$$

Let $\mathcal{P}$ be a property of morphisms stable under base change and composition with isomorphisms. Then whenever $G \to S$ satisfies $\mathcal{P}$, then m satisfies $\mathcal{P}$.