0.1 Group schemes over a field

Let k be a field and $S = \operatorname{Spec} k$.

Lemma 0.1. *Let G be a group scheme over k. Then $G \to \operatorname{Spec} k$ is separated.*

Proof. Let $\pi\colon G \to S$ the structure morphism. Then π is separated if and only if $e\colon S \to G$ is a closed immersion. For any $x \in \operatorname{im}(e) \in G$, choose an affine open neighbourhood $x \in U = \operatorname{Spec} A \subseteq G$. Then $\pi|_U \circ e = \mathrm{id}_S$, hence the induced map $A \xrightarrow{\Gamma(e)} k$ has a section $\Gamma(\pi|_U)$ and is therefore surjective. Thus e is a closed immersion. □

Proposition 0.2. *Let G be a group scheme locally of finite type over k. Then G is smooth over k if and only if G is geometrically reduced.*

Proof. The first direction is immediate, since smoothness is invariant under base change and smooth over a field implies reduced. Conversely, for any field extension ℓ/k by a prior result G is smooth over k if and only if G is smooth over ℓ. Thus we may assume $k = \bar{k}$. By **??** and **??**, we obtain $G_{\mathrm{sm}} \neq \emptyset$. By the transitive action of $G(k)$ on G, every closed point is smooth. Since

$$G_{(0)} = \{g \in G \mid \dim\overline{\{g\}} = 0\}$$

is very dense in G and $G_{\mathrm{sm}} \subseteq G$ is open, the result follows. □

Lemma 0.3. *Let k be perfect and G a group scheme locally of finite type over k. Then the induced reduced subscheme G_{red} is a subgroup scheme of G.*

Proof. Since $(-)_{\mathrm{red}}$ is a functor, we obtain $i\colon G_{\mathrm{red}} \to G_{\mathrm{red}}$ and $e\colon S \to G_{\mathrm{red}}$. By **??**, reduced is equivalent to geometrically reduced since k is perfect. Thus $G_{\mathrm{red}} \times_k G_{\mathrm{red}}$ is reduced and we obtain

$$\begin{array}{ccc} Gx_kG & \xrightarrow{\quad m \quad} & G \\ \uparrow & & \uparrow \\ G_{\mathrm{red}} \times_k G_{\mathrm{red}} & \dashrightarrow & G_{\mathrm{red}} \end{array} .$$

□

Corollary 0.4. *If k is perfect and G a group scheme locally of finite type over k. Then G_{red} is smooth over k.*

Lemma 0.5. *Let G be locally of finite type over k. Then G is geometrically irreducible if (and only if) G is connected.*

Proof. Since $G(k) \neq \emptyset$, we have a morphism $\operatorname{Spec} k \to G$ and $\operatorname{Spec} k$ is geometrically connected. Thus G is geometrically connected. We may therefore assume $k = \bar{k}$. Since the statement is purely topological, we may further assume that G is reduced and thus smooth over k. Hence G is regular by **??**, in particular for every $g \in G$ the local ring $\mathcal{O}_{G,g}$ is regular and hence an integral domain. Since G is locally noetherian and connected, the claim follows. □

Definition 0.6. An *abelian variety* over k is a connected, geometrically reduced and proper k-group scheme.

Remark 0.7. Abelian varieties are smooth and geometrically integral.

Example 0.8. Elliptic curves are abelian varieties of dimension 1.

The goal is now to show that abelian varieties are commutative group schemes.

Lemma 0.9. *Let X be a proper, geometrically connected and geometrically reduced k-scheme and Y an affine k-scheme. Then every morphism $X \xrightarrow{f} Y$ factors over a k-valued point of Y.*

Proof. By the Liouville theorem for schemes, the global sections of $\mathcal{O}_{X_{\bar{k}}}$ is $\bar{k}$. Since $k \to \bar{k}$ is flat, we obtain

$$\Gamma(X, \mathcal{O}_X) \otimes_k \bar{k} \xrightarrow{\simeq} \Gamma(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}).$$

Since $k \to \bar{k}$ is even faithfully flat, we obtain $\Gamma(X, \mathcal{O}_X) \simeq k$.

Choose an embedding $Y \hookrightarrow \mathbb{A}_k^{(I)}$. Then a morphism $f\colon X \to Y$ is equivalent to a morphism $X \xrightarrow{f} Y \hookrightarrow \mathbb{A}_k^{(I)}$, which is equivalent to the datum of a family of $e_i \in \Gamma(X, \mathcal{O}_X)$ which corresponds to a morphism $\operatorname{Spec} k \xrightarrow{e} \mathbb{A}_k^{(I)}$. Thus by construction we obtain a factorisation

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & \mathbb{A}^{(I)} \\ \downarrow & & & \nearrow & \\ \operatorname{Spec} k & & & & \end{array}$$

where the dashed arrow is induced from the isomorphism $\Gamma(X, \mathcal{O}_X) \simeq k$. □

Lemma 0.10 (Rigidity)**.** *Let X be a geometrically reduced, geometrically connected and proper k-scheme with $X(k) \neq \emptyset$. Let further Y be an integral scheme over k, Z be a separated k-scheme and $f\colon X \times_k Y \to Z$ a morphism such that there exists $y \in Y(k)$ such that $f|_{X_y}$ factors via a k-point $z \in Z(k)$. Then f factors via pr_2.*

Proof. Consider the composition

$$g\colon X \times_k Y \xrightarrow{pr_2} Y \simeq \operatorname{Spec} k \times_k Y \xrightarrow{(x_0, \mathrm{id})} X \times_k Y \xrightarrow{f} Z$$

where x_0 is an arbitrarily chosen k-rational point of X. It remains to show that $f = g$. Choose an open affine neighbourhood $z \in U \subseteq Z$. Then $X_y = \mathrm{pr}_2^{-1}(y) \subseteq f^{-1}(U)$. Since X is proper, pr_2 is a closed map. Thus there exists a $y \in V \subseteq Y$ open with $\mathrm{pr}_2^{-1}(V) \subseteq f^{-1}(U)$. For any $y' \in V$, we obtain

$$\begin{array}{ccc} X \times_k Y & \xrightarrow{f} & Z \\ \uparrow & & \uparrow \\ X_{y'} & \dashrightarrow & U \\ {\scriptstyle \alpha(y')}\downarrow & \nearrow & \\ U \times_k \kappa(y') & & \end{array}.$$

By **??**, the morphism $\alpha(y')$ factors over a $\kappa(y')$-valued point. Thus f and g agree on the dense open subset $X \times_k V$. By reduced-to-separated, the result follows. □

Corollary 0.11. *Let A and B be abelian varieties over k and f a morphism of k-schemes $A \to B$. If under the induced map $f(k)\colon A(k) \to B(k)$ the identity e_A is mapped to e_B.*

Proof. Consider the composition

$$g\colon A \times_k A \xrightarrow{(f \circ m_A) \times (i_B \circ m_A \circ (f \times f))} B \times_k B \xrightarrow{m_B} B.$$

It remains to show that the image of g is precisely $\{e_B\}$. By assumption $f(e_A) = e_B$ and thus

$$g(\{e_A\} \times_k A) = \{e_B\} = g(A \times_k \{e_A\}).$$

By repeated application of **??**, g factors via pr_1 and pr_2. Thus g is constant and e_B is in the image. □

Corollary 0.12. *Every abelian variety is commutative.*

Proof. Apply **??** on $i\colon A \to A$. □