0.1 Useful statements on schemes

Let k be a field.

Definition 0.1. Let $\mathcal{P}$ be a property of schemes over fields. For a k-scheme X we say X *is geometrically* $\mathcal{P}$ if for all field extensions K/k the base change $X_K \to \operatorname{Spec} K$ is $\mathcal{P}$.

Example 0.2. The $\mathbb{R}$-scheme $X = \operatorname{Spec}\big(\mathbb{R}[x]/(x^2+1)\big)$ is irreducible but not geometrically irreducible.

Proposition 0.3. *For a k-scheme X the following are equivalent:*

(i) X is geometrically reduced

(ii) for every reduced k-scheme Y, the fibre product $X \times_k Y$ is reduced.

(iii) X is reduced and for every generic point $\eta \in X$ of an irreducible component of X, the field extension $\kappa(\eta)/k$ is separable.

(iv) There exists a perfect field Ω and an extension Ω/k such that X_Ω is reduced.

(v) For all finite and purely inseparable field extensions K/k, the base change X_K is reduced.

Proof. Reducedness is a local property, so without loss of generality $X = \operatorname{Spec} A$. Moreover we may assume that X itself is reduced. Let $\{\eta_i\}_{i\in I}$ be the set of generic points of irreducible components of X. Then we obtain an inclusion

$$A \hookrightarrow \prod_{i\in I} \underbrace{\kappa(\eta_i)}_{=S_i^{-1}A}.$$

We claim that for any field extension L/k the ring $A \otimes_k L$ is reduced if and only if for all $i \in I$ the ring $\kappa(\eta_i) \otimes_k L$ is reduced.

proof of the claim. ($\Rightarrow$): follows since forming the nilradical commutes with localisations. ($\Leftarrow$): We have

$$A \otimes_k L \hookrightarrow \left(\prod_{i\in I} \kappa(\eta_i)\right) \otimes_k L \hookrightarrow \prod_{i\in I} \kappa(\eta_i) \otimes_k L.$$

□

The claim immediatly implies the equivalence of (iii), (iv), (v) and (1). Since (ii) trivially implies (i). It remains to show that (iii) implies (2). Without loss of generality we may take $Y = \operatorname{Spec} B$ and set $\{\lambda_j\}_{j\in J}$ to be the generic points of Y. Then we obtain

$$A \otimes_k B \hookrightarrow A \otimes_k \left(\prod_{j\in J} \kappa(\lambda_j)\right) \hookrightarrow \left(\prod_{i\in I} \kappa(\eta_i)\right) \otimes_k \left(\prod_{j\in J} \kappa(\lambda_j)\right) \hookrightarrow \prod_{i,j} \underbrace{\kappa(\eta_i) \otimes_k \kappa(\eta_j)}_{\text{reduced}}.$$

□

Corollary 0.4. *If k is perfect, then reduced and geometrically reduced are equivalent.*

Remark 0.5. The statements in 0.3 also hold when *reduced* is replaced by *irreducible* or *integral*.

Proposition 0.6. *Let $f\colon X \to Y$ be a morphism of schemes that is locally of finite presentation. Then f is open if and only if for every point $x \in X$ and every point $y' \in Y$ with $y = f(x) \in \overline{\{y'\}}$ there exists $x' \in X$ with $x \in \overline{\{x'\}}$ such that $f(x') = y'$.*

Proof. Assume $X = \text{Spec } B$ and $Y = \text{Spec } A$. ($\Rightarrow$): Then set

$$Z \coloneqq \text{Spec } \mathcal{O}_{X,x} \cap \bigcap_{t \in B \setminus \mathfrak{p}_x} D(t).$$

Since f is open, $y' \in f(D(t))$ for all $t \in B \setminus \mathfrak{p}_x$. Set $f_t \coloneqq f|_{D(t)}$. Then $f_t^{-1}(y') \neq \emptyset$. For sake of contradiction suppose that $y' \notin f(Z)$. Then set $g\colon \text{Spec } \mathcal{O}_{X,x} \to X \xrightarrow{f} Y$. Therefore

$$\emptyset = g^{-1}(y') = \text{Spec } \left(\mathcal{O}_{X,x} \otimes_A \kappa(y')\right).$$

Thus

$$0 = \mathcal{O}_{X,x} \otimes_A \kappa(y') = \text{colim}_{t \in B \setminus \mathfrak{p}_x} \underbrace{B_t \otimes_A \kappa(y')}_{\neq 0}$$

which is a contradiction.

($\Leftarrow$): Show $f(X) \subseteq Y$ is open. By Chevalley's theorem ([?], 10.70), the image $f(X)$ is constructible. In the noetherian case use that open is equivalent to constructible and stable under generalizations ([?], 10.17). In the general case write A as a colimit of noetherian rings and conclude by careful general nonsense. □

Lemma 0.7. *Let $f\colon X \to Y$ be flat, $x \in X$, $y = f(x)$, $y' \in Y$ a generalization of y. Then there exists a generalization x' of x such that $f(x') = y'$.*

Proof. Set $A = \mathcal{O}_{Y,y}$, $B = \mathcal{O}_{X,x}$ and $\varphi\colon A \to B$. Since $y \in \text{im}(f)$ we have $\mathfrak{m}_y B \neq B$ and B is faithfully flat A-module (since φ is local and flat). Thus

$$0 \neq B \otimes_A \kappa(y'),$$

i.e. $f^{-1}(y') \cap \text{Spec } B \neq \emptyset$. □

Corollary 0.8. *Let $f\colon X \to Y$ be flat and locally of finite presentation. Then f is universally open.*

Proof. From 0.6 and 0.7 follows that flat and locally of finite presentation implies open. Since the former two properties are stable under base change, the result follows. □

Corollary 0.9. *Let $f\colon X \to S$ be locally of finite presentation. If $|S|$ is discrete, then every morphism $X \to S$ is universally open.*

Definition 0.10. Let $f\colon X \to Y$. We say

(i) f is *flat in* $x \in X$ if $f_x^{\#}\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

(ii) f is *flat* if f is flat in every point.

Example 0.11. (1) $X \to \text{Spec } k$ is flat.

(2) $\mathbb{A}_Y^n \to Y$ and $\mathbb{P}_Y^n \to Y$ are flat.

(3) Let $f\colon Z \hookrightarrow Y$ be a closed immersion. Then f is flat and locally of finite presentation if and only if f is an open immersion.

Proposition 0.12. *The following holds*

(i) Spec $B \to$ Spec A *is flat if and only if* $A \to B$ *is flat.*

(ii) Flatness is stable under base change and composition.

(iii) Flatness is local on the source and the target.

(iv) Open immersions are flat.

(v) A morphism $f\colon X \to Y$ is flat if and only if for every $y \in Y$ the canonical morphism

$$X \times_Y \operatorname{Spec}(\mathcal{O}_{X,y}) \to \operatorname{Spec}(\mathcal{O}_{Y,y})$$

is flat.

Definition 0.13. A morphism $f\colon X \to Y$ is called *faithfully flat* if f is flat and surjective.

Example 0.14. $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$ is faithfully flat.

Lemma 0.15. *Let $\mathcal{C}$ be a category with equalizers, $F\colon \mathcal{C} \to \mathcal{D}$ a conservative (i.e. reflects isomorphisms) functor that commutes with equalizers. Then F is faithful.*

Proof. Left as an exercise to the reader. □

Proposition 0.16. *Is $f\colon X \to Y$ faithfully flat, then $f^*\colon \operatorname{QCoh}(Y) \to \operatorname{QCoh}(X)$ faithful.*

Proof. Can be deduced from 0.15. The details are left to the reader. □

Remark 0.17 (Faithfully flat descent)**.** The statement from 0.16 can be - from a carefully selected viewpoint - viewn as the statement that the functor $X \mapsto \operatorname{QCoh}(X)$ satisfies the sheaf condition for faithfully flat and quasicompact morphisms, i.e. that the diagram

$$\operatorname{QCoh}(Y) \xrightarrow{f^*} \operatorname{QCoh}(X) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \operatorname{QCoh}(X \times_Y X) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \underbrace{\operatorname{QCoh}(X \times_Y X \times_Y X)}_{\text{corresponds to the cocycle condition}}$$

is a limit diagram.

Proposition 0.18 ([?], 14.53)**.** *Let $f\colon X \to Y$ be a S-morphism and $g\colon S' \to S$ faithfully flat and quasicompact. Denote by $f' = f \times_S S'$. If f' is*

(i) (locally) of finite type or (locally) of finite presentation,

(ii) isomorphism / monomorphism,

(iii) open / closed / quasicompact immersion,

(iv) proper / affine / finite,

then f has the same property.