Lemma 0.1. *Let X be a connected scheme over k and Y a geometrically connected scheme over k. If $\operatorname{Hom}_k(Y, X) \neq \emptyset$, then X is geometrically connected.*

Proof. Use that $X_{\bar{k}} \to X$ is an open and closed immersion. Let $\emptyset \neq Z \subseteq X_{\bar{k}}$ be open and closed. Consider the commutative diagram

$$\begin{array}{ccccc} \bar{f}^{-1}(Z) = Z \times_k Y & \longrightarrow & Y_{\bar{k}} & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ Z & \hookrightarrow & X_{\bar{k}} & \xrightarrow{\pi} & X \end{array}.$$

We obtain $\bar{f}^{-1}(Z) = Y_{\bar{k}}$. Set $Z' = Y_{\bar{k}} \setminus Z$. If Z' is not-empty, then by the same argument $\bar{f}^{-1}(Z') = Y_{\bar{k}}$. Contradiction. □

Proposition 0.2. *Let G be a group scheme locally of finite type over k.*

1. *If $U, V \subseteq G$ are open and dense. Then $UV = G$ as topological spaces.*
2. *If G is irreducible, then G is quasi-compact.*
3. *Any subgroupscheme $H \subseteq G$ is a closed subscheme.*

Proof. We reduce to $k = \bar{k}$.

1. We know that $G_{\bar{k}} \to G$ is an open and closed immersion. Taking pre-images then preserves open and dense (???) and the result follows.
2. By **??** G is geometrically irreducible and $G_{\bar{k}} \to G$ is surjective, i.e. the quasi-compactness of $G_{\bar{k}}$ implies the quasi-compactness of G.
3. By **??**, being a closed immersion can be tested by faithfully flat descent.

Now suppose $k = \bar{k}$.

1. It suffices to show that $U(k)V(k) = G(k)$, since $\overline{U(k)V(k)}$ is very dense in $\overline{UV}$. Since $i \colon G \to G$ is an isomorphism of schemes, $V(k)^{-1} \subseteq G(k)$ is open and dense. Thus for all $g \in G$, $g(V(k)^{-1})$ is open and dense. Thus there exists $u \in g(V(k)^{-1}) \cap U(k)$, i.e. there exists $v \in V(k)$ such that $gv^{-1} = u$, i.e. $g = uv$.
2. Let $U \subseteq G$ be open, dense and quasi-compact. Then $U \times_k U$ is quasi-compact and $G = \operatorname{im}(U \times_k U \to G)$ is quasi-compact.
3. Put the induced reduced subscheme structure on $\bar{H} \subseteq G$. By **??**, the maps $H \to \operatorname{Spec} k$ and $\bar{H} \to \operatorname{Spec} k$ are universally open. Since $H \subseteq \bar{H}$ is dense, we obtain

$$H \times_k H \subseteq H \times_k \bar{H} \subseteq \bar{H} \times_k \bar{H}$$

is dense. Since $H \times_k H \subseteq m^{-1}(H) \subseteq m^{-1}(\bar{H}) \hookrightarrow G \times G$, we obtain topologically $\bar{H} \times \bar{H} \subseteq m^{-1}(\bar{H})$. Since the objects in the lower row are reduced, we therefore obtain a factorisation

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ \uparrow & & \uparrow \\ \bar{H} \times_k \bar{H} & \dashrightarrow & \bar{H} \end{array}.$$

Thus $\bar{H} \subseteq G$ is a subgroupscheme. Thus $H = H \times H = \bar{H}$ where the last equality follows from 1.

□

Definition 0.3. Let G be a group scheme locally of finite type over k and $e\colon \operatorname{Spec} k \to G$ is the unit. Then denote by G^0 the connected component of G that contains $\operatorname{im}(e)$. We call G^0 the *unit component* of G.

Remark 0.4. Since G is locally noetherian, G^0 is open and closed.

Proposition 0.5. *Let G be a group scheme locally of finite type over k.*

1. *G^0 is a quasi-compact, geometrically-irreducible and normal subgroupscheme of G.*
2. *Any group morphism $G \to H$ with H locally of finite type over k induces a group homomorphism $G^0 \to H^0$.*
3. *For any field extension ℓ/k, we have*
$$(G \times_k \ell)^0 = G^0 \times_k \ell.$$

Proof. 1. Since G^0 is connected and contains a k-rational point, by **??** G^0 is geometrically connected. Then $G_0 \times_k G_0$ is connected and

$$\begin{array}{ccc} G \times_k G & \longrightarrow & G \\ \uparrow & & \uparrow \\ G^0 \times_k G^0 & \dashrightarrow & G^0 \end{array}.$$

Since $G^0 \hookrightarrow G \xrightarrow{i} G$ factors over $G^0 \hookrightarrow G$, G^0 is a subgroupscheme. By **??**, G^0 is geometrically irreducible and therefore by **??** it is quasi-compact. For normality consider a connected component G' of G. Then we have a commutative diagram

$$\begin{array}{ccc} G \times_k G & \xrightarrow{(g,h)\mapsto ghg^{-1}} & G \\ \uparrow & & \uparrow \\ G' \times_k G^0 & \dashrightarrow & G^0 \end{array}.$$

Since $G' \times G^0$ is connected, the image of the upper horizontal arrow is in G^0.

2. Any group homomorphism sends the identity to the identity, i.e. the composition $G^0 \hookrightarrow G \to H$ factors via $H^0 \hookrightarrow H$.
3. Since G^0 is geometrically connected, the scheme $G^0 \times_k \ell$ is connected. Moreover $G^0 \times_k \ell \subseteq G \times_k \ell$ is open and closed. Finally, the identity of $G \times_k \ell$ is contained in $G^0 \times_k \ell$ by the universal property of the fibre product.

□

The proof of the following lemma is left as an exercise to the reader.

Lemma 0.6. *Let G be a group scheme locally of finite type over k. Then every connected component of G is quasi-compact and geometrically irreducible and G is equidimensional.*

Proposition 0.7. *Let $f\colon G \to H$ be a group homomorphism of group schemes locally of finite type over k. Then*

1. $\operatorname{im}(f) \subseteq H$ *is closed.*
2. $\dim(G) = \dim(\operatorname{im}(f)) + \dim(\ker(f))$.
3. *Is H smooth over k and f surjective, then f is faithfully flat.*

Remark 0.8. For any integral morphism $f\colon X \to Y$ and $Z \subseteq X$ closed the image $f(Z)$ is closed in Y and $\dim(Z) = \dim(f(Z))$.

Proof. Since $H_{\bar{k}} \xrightarrow{\pi} H$ is integral and surjective and $\dim(Z) = \dim(\pi(Z))$ for any closed subset $Z \subseteq H_{\bar{k}}$, we may assume $k = \bar{k}$.

3. Since smooth implies reduced, H^0 is reduced and by **??** H^0 is irreducible. Thus H^0 is integral. By generic flatness, we have a $V \subseteq H^0$ that is open and dense such that $f^{-1}(V) \to V$ is flat. Thus for all $h \in H(k)$, the map $f^{-1}(hV) \xrightarrow{f} hV$ is flat. By covering H with translates of V, we obtain f is flat.

1. We may assume that G is reduced and thus G is smooth over k by **??**. Let C be $C_{\text{red}} = \overline{f(G)}^H$. We claim that C is a subgroupscheme of H. Then $G \to C$ is quasi-compact and dominant. Thus we have a factorisation

$$\begin{array}{ccccc} G \times_k G & \longrightarrow & C \times_k C & \longrightarrow & H \times_k H \\ \downarrow{\scriptstyle m_G} & & \downarrow{\scriptstyle m_C} & & \downarrow{\scriptstyle m_H} \\ G & \xrightarrow{f} & C & \hookrightarrow & H \end{array}.$$

Analogously one obtains

$$\begin{array}{ccc} C & \dashrightarrow & C \\ \downarrow & & \downarrow \\ H & \longrightarrow & H \end{array}.$$

Thus we may assume that f is dominant. By the theorem of Chevalley, $f(G)$ is constructible and is therefore dense. Hence there exists an open $U \subseteq H$ such that $U \subseteq f(G)$. Thus $H = U \cdot U \subseteq f(G)$ and $f(G) = H$ is closed.

2. We may assume that also H is reduced and that $f(G) = H$. Then H is smooth over k and f is flat. By **??** we have $f(G^0) \subseteq H$ is open and by 1) also closed. Thus $G^0 \xrightarrow{f} H^0$ is surjective. We have $\dim(G^0) = \dim(G)$, $\dim(H^0) = \dim(H)$ and $\dim(\ker(f^0)) = \dim(\ker(f)^0)$. Now the result follows since all fibres are isomorphic and dimension is additive under flat morphism in non-empty fibres ([**?**] 14.119).

□