

# Groupschemes

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# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Useful statements on schemes . . . . .	6

# Chapter 1

## Introduction

Literature: Görtz-Wedhorn: Algebraic Geometry I and II

The goal of this lecture is a brief introduction to the theory of group schemes.

**Definition 1.1** (Group object). Let  $\mathcal{C}$  be a category with finite products. A *group object in  $\mathcal{C}$*  is the data  $(G, m, e, i)$  where

- $G$  is an object of  $\mathcal{C}$
- $m: G \times G \rightarrow G$  is the multiplication map
- $e: 1 \rightarrow G$  is the unit
- $i: G \rightarrow G$  is the inversion map

such that the following diagrams commute

$$\begin{array}{ccc}
 G \times G \times G & \xrightarrow{m \times \text{id}} & G \times G \\
 \downarrow \text{id} \times m & & \downarrow m \\
 G \times G & \xrightarrow{m} & G
 \end{array}
 , \quad
 \begin{array}{ccc}
 G \times G & \xrightarrow{m} & G \\
 \text{id} \times e \uparrow & \swarrow & \\
 G \times 1 & & 
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 G & \xrightarrow{\text{id} \times i} & G \times G \\
 \downarrow & & \downarrow m \\
 1 & \xrightarrow{e} & G
 \end{array}
 .$$

$G$  is called *commutative*, if additionally the diagram

$$\begin{array}{ccc}
 G \times G & \xrightarrow{\text{swap}} & G \times G \\
 \downarrow m & \swarrow m & \\
 G & & 
 \end{array}$$

commutes.

A *morphism of group objects*  $(G, m, e, i) \rightarrow (G', m', e', i')$  is a morphism  $f: G \rightarrow G'$  in  $\mathcal{C}$  such that the diagrams

$$\begin{array}{ccc}
 G \times G & \xrightarrow{f \times f} & G' \times G' \\
 \downarrow m & & \downarrow m' \\
 G & \xrightarrow{f} & G'
 \end{array}
 , \quad
 \begin{array}{ccc}
 G & \xrightarrow{f} & G' \\
 e \uparrow & \swarrow e' & \\
 1 & & 
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 G & \xrightarrow{f} & G' \\
 \downarrow i & & \downarrow i' \\
 G & \xrightarrow{f} & G'
 \end{array}
 .$$

This defines the category  $\text{Grp}(\mathcal{C})$  of group objects in  $\mathcal{C}$ .

**Example 1.2.** •  $\text{Grp}(\text{Set}) \simeq \text{Grp}$

- $\text{Grp}(\text{Grp}) \simeq \text{Ab}$
- $\text{Grp}(\text{Ab}) \simeq ?$

- $\text{Grp}(\text{Top}) \simeq$  topological Groups
- $\text{Grp}(\text{Mfd}_\infty) \simeq$  Lie Groups

**Definition 1.3** (group scheme). Let  $S$  be a scheme. An  $S$ -group scheme or an  $S$ -group is a group object in the category of schemes over  $S$ .

**Remark 1.4.** Let  $S$  be a scheme. The structure of a group scheme over  $S$  on a  $S$ -scheme  $G$  is equivalent to a factorisation of the functor of points

$$\begin{array}{ccc} \text{Sch}_S & \longrightarrow & \text{Set} \\ \downarrow & \nearrow & \\ \text{Grp} & & \end{array}$$

via the forgetful functor from groups to sets.

**Example 1.5.** Let  $S$  be a scheme.

- Let  $\Gamma$  be a group. Then  $G = \Gamma_S$  where  $G(T) := \{ \text{locally constant maps } T \rightarrow \Gamma \}$
- (additive group)  $\mathbb{G}_{a,S}$  where  $\mathbb{G}_{a,S}(T) = \mathcal{O}_T(T)$ . We have  $\mathbb{G}_{a,S} \simeq \mathbb{A}_S^1$ .
- (multiplicative group)  $\mathbb{G}_{m,S}$  where  $\mathbb{G}_{m,S}(T) := \mathcal{O}_T(T)^\times$ .
- (roots of unity)  $\mu_{n,S}$  ( $n \geq 1$ ) where  $\mu_{n,S}(T) = \{x \in \mathcal{O}_T(T)^\times \mid x^n = 1\}$ .
- $S = \text{Spec}(R)$ .  $\text{GL}_{n,R} = \text{Spec}(A)$  where  $A = R[T_{ij} \mid 1 \leq i, j \leq n][\det^{-1}]$  where  $\det = \sum_{\sigma \in S_n} \text{sgn}(\sigma) T_{1\sigma(1)} \cdots T_{n\sigma(n)}$ . We obtain  $\text{GL}_{n,S}$  by base changing  $\text{GL}_{n,\mathbb{Z}}$ .

**Lemma 1.6.** Let  $G$  be a  $S$ -group. Then  $G \rightarrow S$  is separated if and only if  $S \xrightarrow{e} G$  is a closed immersion.

**Definition 1.7.** Let  $R$  be a ring. A (commutative) Hopf-Algebra over  $R$  is a group object in  $\text{Alg}_R^{\text{op}}$ , where  $\text{Alg}_R = \text{CRing}_R$ .

**Remark 1.8.** For a  $R$ -Hopf-Algebra  $A$ , we denote the canonical maps by

- $\mu: A \rightarrow A \otimes_R A$  (Comultiplication)
- $\varepsilon: A \rightarrow R$  (Counit)
- $\iota: A \rightarrow A$  (Antipode)

A Hopf-Algebra is called *cocommutative*, if the associated group object in  $\text{Alg}_R^{\text{op}}$  kommutativ ist.

**Remark 1.9.** For a ring  $R$ , by construction we have an equivalence of categories between the category of affine  $R$ -group schemes and the opposite category of  $R$ -Hopf-Algebras.

**Example 1.10.** The additive group  $\mathbb{G}_{a,R} = \text{Spec}(R[t])$  has

- comultiplication  $\mu: R[t] \rightarrow R[t] \otimes_R R[t], t \mapsto t \otimes 1 - 1 \otimes t$ .
- counit  $\varepsilon: R[t] \rightarrow R, t \mapsto 0$
- antipode  $\iota: R[t] \rightarrow R[t], t \mapsto -t$

*Proof.* For any  $R$ -algebra  $A$  we have  $\mathbb{G}_{a,R}(A) = A$  and the diagram

$$\begin{array}{ccc} \mathbb{G}_{a,R}(A) \times \mathbb{G}_{a,R}(A) & \xrightarrow{m} & \mathbb{G}_{a,R}(A) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Hom}_R(R[s_1, s_2], A) & \xrightarrow{\mu^*} & \text{Hom}_R(R[t], A) \end{array} .$$

□

**Definition 1.11.** Let  $G$  be a  $S$ -group. A *subgroupscheme* of  $G$  is a subscheme  $H \subseteq G$  such that

- 1) for all  $T \in \text{Sch}_S$ , we have  $H(T) \subseteq G(T)$  a subgroup,
- 2) We have commutative diagrams

$$\begin{array}{ccc}
 H \times_S H & \longrightarrow & G \times_S G \xrightarrow{m} G \\
 \downarrow & \nearrow & \\
 H & & 
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 S & \xrightarrow{e} & G \\
 \downarrow & \nearrow & \\
 H & & 
 \end{array}$$

A subgroup scheme  $H \subseteq G$  is *normal* if  $H(T)$  is a normal subgroup of  $G(T)$  for all  $T \in \text{Sch}_S$ .

For a morphism  $f: G \rightarrow G'$  of  $S$ -groups and a subgroup  $H' \subseteq G'$ , let  $f^{-1}(H')$  be  $G \times'_G H'$ . For  $H' = 1 \xrightarrow{e} G'$ , we obtain the *kernel of  $f$*  and the cartesian square

$$\begin{array}{ccc}
 \text{Ker}(f) & \longrightarrow & G \\
 \downarrow & & \downarrow f \\
 S & \xrightarrow{e} & G'
 \end{array}$$

**Remark 1.12.** The kernel of a homomorphism  $f$  of  $S$ -groups is for any  $S$ -scheme  $T$  given by

$$\text{Ker}(f)(T) = \ker(f(T)).$$

In particular, the  $\text{Ker}(f)$  is normal.

**Definition 1.13.** Let  $G$  be a  $S$ -group,  $T$  a  $S$ -scheme and  $g \in G(T) = \text{Hom}_S(T, G)$ . Define the *lefttranslation by  $g$*  as

$$\begin{array}{ccc}
 G_T & \xleftarrow{=} & T \times_T G_T \\
 \downarrow t_g & & \downarrow g \times \text{id} \\
 G_T & \xleftarrow{m} & G_T \times_T G_T
 \end{array}$$

**Remark 1.14.** In the situation of 1.13, for every  $T' \xrightarrow{f} T$ , the map

$$t_g(T'): G_T(T') = G(T') \longrightarrow G(T') = G_T(T')$$

is the lefttranslation by the element  $f^*(g) \in G(T')$ .

**Remark 1.15.** Consider

$$\begin{array}{ccc}
 G \times_S G & \xrightarrow{(g,h) \mapsto (gh,h)} & G \times_S G \\
 \downarrow m & \nearrow \text{pr}_1 & \\
 G & & 
 \end{array}$$

Let  $\mathcal{P}$  be a property of morphisms stable under base change and composition with isomorphisms. Then whenever  $G \rightarrow S$  satisfies  $\mathcal{P}$ , then  $m$  satisfies  $\mathcal{P}$ .

## 1.1 Useful statements on schemes

Let  $k$  be a field.

**Definition 1.16.** Let  $\mathcal{P}$  be a property of schemes over fields. For a  $k$ -scheme  $X$  we say  $X$  is *geometrically  $\mathcal{P}$*  if for all field extensions  $K/k$  the base change  $X_K \rightarrow \text{Spec } K$  is  $\mathcal{P}$ .

**Example 1.17.** The  $\mathbb{R}$ -scheme  $X = \text{Spec}(\mathbb{R}[x]/(x^2 + 1))$  is irreducible but not geometrically irreducible.

**Proposition 1.18.** *For a  $k$ -scheme  $X$  the following are equivalent:*

- (i)  $X$  is geometrically reduced
- (ii) for every reduced  $k$ -scheme  $Y$ , the fibre product  $X \times_k Y$  is reduced.
- (iii)  $X$  is reduced and for every generic point  $\eta \in X$  of an irreducible component of  $X$ , the field extension  $\kappa(\eta)/k$  is separable.
- (iv) There exists a perfect field  $\Omega$  and an extension  $\Omega/k$  such that  $X_\Omega$  is reduced.
- (v) For all finite and purely inseparable field extensions  $K/k$ , the base change  $X_K$  is reduced.

*Proof.* Reducedness is a local property, so without loss of generality  $X = \text{Spec } A$ . Moreover we may assume that  $X$  itself is reduced. Let  $\{\eta_i\}_{i \in I}$  be the set of generic points of irreducible components of  $X$ . Then we obtain an inclusion

$$A \hookrightarrow \prod_{i \in I} \underbrace{\kappa(\eta_i)}_{=S_i^{-1}A}.$$

We claim that for any field extension  $L/k$  the ring  $A \otimes_k L$  is reduced if and only if for all  $i \in I$  the ring  $\kappa(\eta_i) \otimes_k L$  is reduced.

*proof of the claim.* ( $\Rightarrow$ ): follows since forming the nilradical commutes with localisations. ( $\Leftarrow$ ): We have

$$A \otimes_k L \hookrightarrow \left( \prod_{i \in I} \kappa(\eta_i) \right) \otimes_k L \hookrightarrow \prod_{i \in I} \kappa(\eta_i) \otimes_k L.$$

□

The claim immediately implies the equivalence of (iii), (iv), (v) and (1). Since (ii) trivially implies (i). It remains to show that (iii) implies (2). Without loss of generality we may take  $Y = \text{Spec } B$  and set  $\{\lambda_j\}_{j \in J}$  to be the generic points of  $Y$ . Then we obtain

$$A \otimes_k B \hookrightarrow A \otimes_k \left( \prod_{j \in J} \kappa(\lambda_j) \right) \hookrightarrow \left( \prod_{i \in I} \kappa(\eta_i) \right) \otimes_k \left( \prod_{j \in J} \kappa(\lambda_j) \right) \hookrightarrow \prod_{i,j} \underbrace{\kappa(\eta_i) \otimes_k \kappa(\lambda_j)}_{\text{reduced}}.$$

□

**Corollary 1.19.** *If  $k$  is perfect, then reduced and geometrically reduced are equivalent.*

**Remark 1.20.** The statements in 1.18 also hold when *reduced* is replaced by *irreducible* or *integral*.

**Proposition 1.21.** *Let  $f: X \rightarrow Y$  be a morphism of schemes that is locally of finite presentation. Then  $f$  is open if and only if for every point  $x \in X$  and every point  $y' \in Y$  with  $y = f(x) \in \{y'\}$  there exists  $x' \in X$  with  $x \in \overline{\{x'\}}$  such that  $f(x') = y'$ .*

*Proof.* Assume  $X = \text{Spec } B$  and  $Y = \text{Spec } A$ . ( $\Rightarrow$ ): Then set

$$Z := \text{Spec } \mathcal{O}_{X,x} \cap \bigcap_{t \in B \setminus \mathfrak{p}_x} D(t).$$

Since  $f$  is open,  $y' \in f(D(t))$  for all  $t \in B \setminus \mathfrak{p}_x$ . Set  $f_t := f|_{D(t)}$ . Then  $f_t^{-1}(y') \neq \emptyset$ . For sake of contradiction suppose that  $y' \notin f(Z)$ . Then set  $g: \text{Spec } \mathcal{O}_{X,x} \rightarrow X \xrightarrow{f} Y$ . Therefore

$$\emptyset = g^{-1}(y') = \text{Spec } (\mathcal{O}_{X,x} \otimes_A \kappa(y')).$$

Thus

$$0 = \mathcal{O}_{X,x} \otimes_A \kappa(y') = \text{colim}_{t \in B \setminus \mathfrak{p}_x} \underbrace{B_t \otimes_A \kappa(y')}_{\neq 0}$$

which is a contradiction.

( $\Leftarrow$ ): Show  $f(X) \subseteq Y$  is open. By Chevalley's theorem ([GW10], 10.70), the image  $f(X)$  is constructible. In the noetherian case use that open is equivalent to constructible and stable under generalizations ([GW10], 10.17). In the general case write  $A$  as a colimit of noetherian rings and conclude by careful general nonsense.  $\square$

**Lemma 1.22.** *Let  $f: X \rightarrow Y$  be flat,  $x \in X$ ,  $y = f(x)$ ,  $y' \in Y$  a generalization of  $y$ . Then there exists a generalization  $x'$  of  $x$  such that  $f(x') = y'$ .*

*Proof.* Set  $A = \mathcal{O}_{Y,y}$ ,  $B = \mathcal{O}_{X,x}$  and  $\varphi: A \rightarrow B$ . Since  $y \in \text{im}(f)$  we have  $\mathfrak{m}_y B \neq B$  and  $B$  is faithfully flat  $A$ -module (since  $\varphi$  is local and flat). Thus

$$0 \neq B \otimes_A \kappa(y'),$$

i.e.  $f^{-1}(y') \cap \text{Spec } B \neq \emptyset$ .  $\square$

**Corollary 1.23.** *Let  $f: X \rightarrow Y$  be flat and locally of finite presentation. Then  $f$  is universally open.*

*Proof.* From 1.21 and 1.22 follows that flat and locally of finite presentation implies open. Since the former two properties are stable under base change, the result follows.  $\square$

**Corollary 1.24.** *Let  $f: X \rightarrow S$  be locally of finite presentation. If  $|S|$  is discrete, then every morphism  $X \rightarrow S$  is universally open.*

**Definition 1.25.** Let  $f: X \rightarrow Y$ . We say

- (i)  $f$  is flat in  $x \in X$  if  $f_x^\#: \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$  is flat.
- (ii)  $f$  is flat if  $f$  is flat in every point.

**Example 1.26.** (1)  $X \rightarrow \text{Spec } k$  is flat.

(2)  $\mathbb{A}_Y^n \rightarrow Y$  and  $\mathbb{P}_Y^n \rightarrow Y$  are flat.

(3) Let  $f: Z \hookrightarrow Y$  be a closed immersion. Then  $f$  is flat and locally of finite presentation if and only if  $f$  is an open immersion.

**Proposition 1.27.** *The following holds*

- (i)  $\text{Spec } B \rightarrow \text{Spec } A$  is flat if and only if  $A \rightarrow B$  is flat.
- (ii) Flatness is stable under base change and composition.
- (iii) Flatness is local on the source and the target.

(iv) *Open immersions are flat.*

(v) *A morphism  $f: X \rightarrow Y$  is flat if and only if for every  $y \in Y$  the canonical morphism*

$$X \times_Y \operatorname{Spec}(\mathcal{O}_{X,y}) \rightarrow \operatorname{Spec}(\mathcal{O}_{Y,y})$$

*is flat.*

**Definition 1.28.** A morphism  $f: X \rightarrow Y$  is called *faithfully flat* if  $f$  is flat and surjective.

**Example 1.29.**  $\operatorname{Spec} \bar{k} \rightarrow \operatorname{Spec} k$  is faithfully flat.

**Lemma 1.30.** *Let  $\mathcal{C}$  be a category with equalizers,  $F: \mathcal{C} \rightarrow \mathcal{D}$  a conservative (i.e. reflects isomorphisms) functor that commutes with equalizers. Then  $F$  is faithful.*

*Proof.* Left as an exercise to the reader. □

**Proposition 1.31.** *Is  $f: X \rightarrow Y$  faithfully flat, then  $f^*: \operatorname{QCoh}(Y) \rightarrow \operatorname{QCoh}(X)$  faithful.*

*Proof.* Can be deduced from 1.30. The details are left to the reader. □

**Remark 1.32** (Faithfully flat descent). The statement from 1.31 can be - from a carefully selected viewpoint - viewn as the statement that the functor  $X \mapsto \operatorname{QCoh}(X)$  satisfies the sheaf condition for faithfully flat and quasicompact morphisms, i.e. that the diagram

$$\operatorname{QCoh}(Y) \xrightarrow{f^*} \operatorname{QCoh}(X) \begin{array}{c} \xrightarrow{\operatorname{pr}_1^*} \\ \xrightarrow{\operatorname{pr}_2^*} \end{array} \operatorname{QCoh}(X \times_Y X) \rightrightarrows \underbrace{\operatorname{QCoh}(X \times_Y X \times_Y X)}_{\text{corresponds to the cocycle condition}}$$

is a limit diagram.

**Proposition 1.33** ([GW10], 14.53). *Let  $f: X \rightarrow Y$  be a  $S$ -morphism and  $g: S' \rightarrow S$  faithfully flat and quasicompact. Denote by  $f' = f \times_S S'$ . If  $f'$  is*

- (i) *(locally) of finite type or (locally) of finite presentation,*
- (ii) *isomorphism / monomorphism,*
- (iii) *open / closed / quasicompact immersion,*
- (iv) *proper / affine / finite,*

*then  $f$  has the same property.*

# Bibliography

- [GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.