1 Overview

These notes mostly follow [Mat]. Some ideas are taken from [GM03].

In the following, for a topological space X denote by $\mathcal{A}b(X)$ the category of sheaves of abelian groups on X. Furthermore, denote by $\mathrm{D}^+(X)$ the bounded below derived category of $\mathcal{A}b(X)$.

Definition 1.1 (Lower Shriek)**.** Let $f\colon X \to Y$ be a continuous map of locally compact topological spaces. For $\mathcal{F} \in \mathcal{A}b(X)$ and $U \subseteq Y$ open, let

$$f_!(\mathcal{F})(U) = \{s \in \mathcal{F}(f^{-1}(U))\colon \operatorname{supp}(s) \xrightarrow{f} U \text{ proper}\}.$$

Lemma 1.2 (Lower shriek of sheaf is a sheaf)**.** *Let $\mathcal{F} \in \mathcal{A}b(X)$ be a sheaf $f\colon X \to Y$ continuous. Then $f_!\mathcal{F}$ is a sheaf on Y.*

Proof. Clearly, $f_!\mathcal{F}$ is a sub-presheaf of the sheaf $f_*\mathcal{F}$. To show it is a sheaf, we need to verify that gluing sections in $f_!\mathcal{F}$ gives again a section in $f_!\mathcal{F}$.

Let $(U_i)_{i\in I}$ be a family of open sets in Y and $s_i \in (f_!\mathcal{F})(U_i)$ sections. Thus $s_i \in \mathcal{F}(f^{-1}(U_i))$ such that $\operatorname{supp}(s_i) \xrightarrow{f} U_i$ is proper. Gluing yields a unique section $s \in \mathcal{F}(f^{-1}(U))$. We need to check that

$$\operatorname{supp}(s) = \bigcup_{i\in I} \operatorname{supp}(s_i) \xrightarrow{f} \bigcup_{i\in I} U_i$$

is proper. For this note that $\left(f|_{\operatorname{supp}(s)}\right)^{-1}(U_i) = f^{-1}(U_i) \cap \operatorname{supp}(s) = \operatorname{supp}(s_i)$ and being proper is local on the target. □

The goal of this and the following talk is to prove the following theorem

Theorem 1.3 (Verdier duality)**.** *If X, Y are locally compact topological spaces of finite dimension, then $\mathrm{R}f_!$ admits a right adjoint $f^!\colon \mathrm{D}^+(Y) \to \mathrm{D}(X)$.*

To show the existence of the derivative of $f_!$, we need to introduce an adapted class of shaves.

Definition 1.4. Let X be a locally compact space, $\mathcal{F} \in \mathcal{A}b(X)$ and $Z \subseteq X$ a subset. Then define

$$\mathcal{F}(Z) = \Gamma(Z, \mathcal{F}) = \Gamma(Z, i^*\mathcal{F})$$

for $i\colon Z \to X$ the canonical inclusion.

Remark 1.5. If $Z \subseteq X$ is a subset and $i\colon Z \to X$ the canonical inclusion, then

$$\mathcal{F}(Z) = \left\{ (s_i, U_i)_{i\in I} \colon U_i \subseteq X \text{ open with } Z \subseteq \bigcup_{i\in I} U_i, s_i \in \mathcal{F}(U_i) \text{ with } (s_i)_z = (s_j)_z \forall i,j \in I, z \in Z \cap U_i \cap U_j \right\} / \sim .$$

where $(U_i, s_i)_{i\in I} \sim (V_j, t_j)_{j\in J}$ if and only if $(s_i)_z = (t_j)_z$ for all $i \in I$, $j \in J$ and $z \in U_i \cap V_j \cap Z$.

For every open neighbourhood U of Z, we have a restriction map

$$\mathcal{F}(U) \to \mathcal{F}(Z), s \mapsto s|_Z \coloneqq [(s, U)].$$

This induces a map

$$\operatorname*{colim}_{Z\subseteq U} \mathcal{F}(U) \to \mathcal{F}(Z).$$

Lemma 1.6. *Let X be a locally compact Hausdorff space and $\mathcal{F} \in \mathcal{A}b(X)$. If $Z \subseteq X$ is compact, the natural map*

$$\operatorname*{colim}_{Z\subseteq U} \mathcal{F}(U) \longrightarrow \mathcal{F}(Z)$$

is an isomorphism.

Proof. Injectivity: Let $s \in \mathcal{F}(U)$ such that $s|_Z = 0$. Thus for all $z \in Z$, $s_z = 0$ and there exists an open neighbourhood $z \in U_z \subseteq U$ such that $s|_{U_z} = 0$. Thus $s|_{\bigcup U_z} = 0$. Since $Z \subseteq \bigcup_{z\in Z} U_z$, s is zero in the colimit.

Surjectivity: Take $(s_i, U_i)_{i\in I} \in \mathcal{F}(Z)$. Thus $Z \subseteq \bigcup_{i\in I} U_i$ and by local compactness, for every $z \in Z$, there exists a compact neighbourhood $z \in K_z$ such that $K_z \subseteq U_{i_z}$ for some $i_z \in I$. Since Z is compact, finitely many suffice, so we may assume $Z \subseteq \bigcup_{i=1}^n K_i$ and $K_i \subseteq U_i \subseteq X$. We now want to define a section on a neighbourhood of Z that locally agrees with the s_i.

By induction, we may assume $n = 2$. By definition, $(s_1)_z = (s_2)_z$ for all $z \in Z \cap U_1 \cap U_2$, in particular $s_1|_{U_1\cap U_2}$ and $s_2|_{U_1\cap U_2}$ have the same restriction to $K_1 \cap K_2$. By the injectivity of the restriction map, there exists an open neighbourhood $K_1 \cap K_2 \subseteq V \subseteq U_1 \cap U_2$, such that $s_1|_V = s_2|_V$. Since $K_j \setminus V$ is closed in the compact K_j, for $j = 1, 2$ the subset $K_j \setminus V$ is compact. Since X is Hausdorff, there exist open neighbourhoods $K_j \setminus V \subseteq U_j' \subseteq U_j$ such that $U_1' \cap U_2' = \emptyset$. Now $s_1|_{U_1'}$, $s_2|_{U_2'}$ and $s_1|_V = s_2|_V$ glue to a section w on $U_1' \cup U_2' \cup V \supseteq K_1 \cup K_2 \supseteq Z$ such that $w|_Z = [(s_i, U_i)_{i\in I}]$. □

Definition 1.7. A sheaf $\mathcal{F} \in \mathcal{A}b(X)$ is ***soft*** if $\mathcal{F}(X) \to \mathcal{F}(Z)$ is surjective whenever $Z \subseteq X$ is compact.

Remark 1.8. In [KS94] our notion of softness is called *c-soft.* For σ-compact spaces the notions agree according to Exercise II.6 in [KS94].

Remark 1.9 (Flasque sheaves are soft)**.** Recall that a sheaf $\mathcal{F} \in \mathcal{A}b(X)$ is called *flasque*, if for every open set $U \subseteq X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. For $Z \subseteq X$ compact, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(Z) \\ & \searrow\!\!\!\!\searrow \quad \nearrow_{\simeq} & \\ & \operatorname*{colim}_{Z\subseteq U} \mathcal{F}(U) & \end{array} .$$

Thus $\mathcal{F}$ is soft.

Proposition 1.10. *Let X be a locally compact topological space. If $\mathcal{F} \in \mathcal{A}b(X)$ is soft, $K \subseteq X$ is compact and $K \subseteq U$ is an open neighbourhood, any section over K can be extended to a global section with compact support contained in U.*

Proof. Let $s \in \mathcal{F}(K)$. By local compactness, there exists a compact neighbourhood L of K with $L \subseteq U$. Then $K \cap \partial L = \emptyset$. Consider the section on $K \cup \partial L$ given by s on K and zero on ∂L. Since $\mathcal{F}$ is soft, this can be extended to a global section, and a fortiori to a section t over L. Now the sections given by t on L and 0 on $\overline{X \setminus L}$ glue to a compactly supported extension of s. Since $L \subseteq U$, its support is contained in U. □

1.1 Compactly supported cohomology

Let X be a topological space.

Remark 1.11 (Support)**.** For $\mathcal{F} \in \mathcal{A}b(X)$, $U \subseteq X$ open and a section $s \in \mathcal{F}(U)$, its support $\operatorname{supp}(s)$ is defined as

$$\{x \in U : s_x \neq 0\}.$$

This set is always closed, as its complement is open.

Definition 1.12. Let $U \subseteq X$ be open and $\mathcal{F} \in \mathcal{A}b(X)$. We define $\Gamma_c(U, \mathcal{F})$ as the subgroup of $\Gamma(U, \mathcal{F})$ consisting of sections with compact support.

Remark 1.13. If $s, t \in \Gamma(U, \mathcal{F})$ have compact support, so does $s + t$. Thus $\Gamma_c(U, \mathcal{F})$ is indeed a subgroup of $\Gamma(U, \mathcal{F})$.

Taking $U = X$, this defines a functor $\Gamma_c = \Gamma_c(X, \cdot) : \mathcal{A}b(X) \to \mathcal{A}b$

Remark 1.14 (Lower shriek and compact support)**.** Let $f\colon X \to \{*\}$ be the unique continuous map from X to the one point space. Then $f_! \cdot = \Gamma_c(X, \cdot)$

Proposition 1.15. *Γ_c is left exact.*

Proof. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ be an exact sequence in $\mathcal{A}b(X)$. This induces a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(X,\mathcal{F}') & \longrightarrow & \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{F}'') \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Gamma_c(X,\mathcal{F}') & \longrightarrow & \Gamma_c(X,\mathcal{F}) & \longrightarrow & \Gamma_c(X,\mathcal{F}'') \end{array},$$

where the first row is exact. Since the vertical arrows are inclusions, the injectivity of $\Gamma_c(X, \mathcal{F}') \to \Gamma_c(X, \mathcal{F})$ is immediate. Let now $s \in \Gamma_c(X, \mathcal{F}) \subseteq \Gamma(X, \mathcal{F})$ such that s becomes zero in $\Gamma_c(X, \mathcal{F}'')$. Thus by exactness of the first row, $s \in \Gamma(X, \mathcal{F}')$. Since $s \in \Gamma_c(X, \mathcal{F})$, s is compactly supported, so $s \in \Gamma_c(X, \mathcal{F}')$. □

Proposition 1.16. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence in $\mathcal{A}b(X)$. Suppose $\mathcal{F}'$ is soft. Then the sequence $0 \to \Gamma_c(X, \mathcal{F}') \to \Gamma_c(X, \mathcal{F}) \to \Gamma_c(X, \mathcal{F}'') \to 0$ is also exact.*

Proof. By 1.15, we only need to show surjectivity on the right.

Suppose first that X is compact and let $s \in \Gamma_c(X, \mathcal{F}'') = \Gamma(X, \mathcal{F}'')$. Since $\mathcal{F} \to \mathcal{F}'' \to 0$ is exact, there exist a covering $X = \bigcup_{i \in I} U_i$ and lifts $t_i \in \mathcal{F}(U_i)$ of $s|_{U_i}$. By local compactness of X, we may assume, after a possible refinement, that each U_i contains a compact set V_i whose interiors still cover X. Since X is compact, we may assume I is finite. To piece together the t_i, we may assume, by induction, that $\#I = 2$.

Consider $t_1|_{U_1 \cap U_2} - t_2|_{U_1 \cap U_2}$. This is necessarily a section e' of $\mathcal{F}'(U_1 \cap U_2)$ as it maps to zero in $\mathcal{F}''(U_1 \cap U_2)$. Restricting e' to the compact $V_1 \cap V_2$ and extending it by softness, yields a global section e of $\mathcal{F}'$. Now

$$(t_2|_{V_2} + e|_{V_2})|_{V_1 \cap V_2} = t_2|_{V_1 \cap V_2} + e'|_{V_1 \cap V_2} = t_1|_{V_1 \cap V_2}.$$

Thus $t_1|_{V_1}, t_2|_{V_2} + e|_{V_2}$ glue to a global section t of $\mathcal{F}$ with image s.

Now for general X: Let $s \in \mathcal{F}''(X)$ with compact support Z. By local compactness, there exists a compact neighbourhood $Z' \subseteq X$ of Z. Since pullback of sheaves is exact and restriction of soft sheaves to closed subsets preserves softness, applying the result to Z', yields a section $t' \in \mathcal{F}(Z')$ lifting $s|_{Z'}$. The restriction $t'|_{\partial Z'}$ maps to $s|_{\partial Z'} = 0$, so $t'|_{\partial Z'} \in \mathcal{F}'(\partial Z')$. Since $\partial Z'$ is compact and $\mathcal{F}'$ is soft, $t'|_{\partial Z'}$ extends to a global section b of $\mathcal{F}'$. Thus

$$(t' - b|_{Z'})|_{\partial Z'} = t'|_{\partial Z'} - t'|_{\partial Z'} = 0.$$

So $t' - b|_{Z'}$ on Z' and 0 on $\overline{X \setminus Z'}$ glue to a global section t of $\mathcal{F}$. Then $t|_{Z'} = t' - b|_{Z'}$ maps to $s|_{Z'}$ since $b \in \mathcal{F}'(X)$. Since $\operatorname{supp}(t), \operatorname{supp}(s) \subseteq Z'$, t is a compactly supported lift of s. □

Corollary 1.17. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence in $\mathcal{A}b(X)$ and $\mathcal{F}', \mathcal{F}$ are soft, then $\mathcal{F}''$ is soft too.*

Proof. Let $Z \subseteq X$ be compact. Since restricting to a closed subset is exact and preserves softness, by 1.16 $\Gamma_c(Z, \mathcal{F}) \to \Gamma_c(Z, \mathcal{F}'')$ is surjective. This yields a commutative diagram

$$\begin{array}{ccc} \Gamma_c(X,\mathcal{F}) & \longrightarrow & \Gamma_c(X,\mathcal{F}'') \\ \downarrow\!\!\!\!\downarrow & & \downarrow \\ \Gamma_c(Z,\mathcal{F}) & \twoheadrightarrow & \Gamma_c(Z,\mathcal{F}'') \end{array},$$

where the left vertical arrow is surjective, since $\mathcal{F}$ is soft. Since the composition is surjective, $\Gamma_c(X, \mathcal{F}'') \to \Gamma_c(Z, \mathcal{F}'')$ is also surjective. □

Corollary 1.18. *Soft sheaves are Γ_c-acyclic.*

Proof. Let $\mathcal{F} \in \mathcal{A}b(X)$ be soft and embed $\mathcal{F}$ in an injective sheaf $\mathcal{I}$. This yields an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0 \ .$$

Since $\mathcal{I}$ is injective, in particular flasque, hence soft, by 1.17, $\mathcal{G}$ is soft. We proceed by induction. For $i = 1$ consider the exact sequence

$$0 \longrightarrow \Gamma_c(X, \mathcal{F}) \longrightarrow \Gamma_c(X, \mathcal{I}) \longrightarrow \Gamma_c(X, \mathcal{G}) \longrightarrow H^1_c(X, \mathcal{F}) \longrightarrow \underbrace{H^1_c(X, \mathcal{I})}_{=0} \ .$$

Since $\mathcal{F}$ is soft, $\Gamma_c(X, \mathcal{I}) \to \Gamma_c(X, \mathcal{G})$ is surjective. By the exactness of the sequence, $H^1_c(X, \mathcal{F})$ vanishes. Now assume $H^i_c(X, \mathcal{F}) = 0$ for any soft sheaf $\mathcal{F}$. Then the exact sequence

$$\underbrace{H^i_c(X, \mathcal{I})}_{=0} \longrightarrow H^i_c(X, \mathcal{G}) \longrightarrow H^{i+1}_c(X, \mathcal{F}) \longrightarrow \underbrace{H^{i+1}_c(X, \mathcal{I})}_{=0}$$

yields an isomorphism $H^i_c(X, \mathcal{G}) \simeq H^{i+1}_c(X, \mathcal{F})$ and since $\mathcal{G}$ is soft, the left hand side is zero by induction hypothesis. ☐

Theorem 1.19. *Let $f \colon X \to Y$ be a continuous map of locally compact topological spaces. If Y is Hausdorff and $\mathcal{F} \in \mathcal{A}b(X)$, then there is a natural isomorphism*

$$(R^i f_! \mathcal{F})_y \simeq H^i_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$$

for each $y \in Y$.

Proof. Denote by X_y the fibre of f over y and by $\mathcal{F}$ the restriction to X_y. Let $y \in Y$. Since $R^i f_!$ is a derived functor, it is a universal δ-functor. Since restriction of soft sheaves to closed subspaces preserves softness, the δ-functor $\mathcal{F} \mapsto H^i_c(X_y, \mathcal{F}_y)$ vanishes for soft sheaves and $i > 0$. Thus it is effaceable and hence universal. Therefore it suffices to define a natural isomorphism in degree 0.

Let $y \in U \subseteq Y$ open. Then consider the natural map

$$\begin{aligned} (f_! \mathcal{F})(U) &\longrightarrow \Gamma_c(X_y, \mathcal{F}_y) \\ s &\longmapsto s|_{X_y}. \end{aligned}$$

This is well-defined, since for any $s \in \mathcal{F}(f^{-1}(U))$ with $\mathrm{supp}(s) \xrightarrow{f} U$ proper, we have

$$\mathrm{supp}(s|_{X_y}) = \mathrm{supp}(s) \cap X_y = \left(f|^U_{\mathrm{supp}(s)} \right)^{-1} (y)$$

and the right hand side is compact. This map induces a natural map

$$(f_! \mathcal{F})_y = \operatorname*{colim}_{y \in U \subseteq Y} (f_! \mathcal{F})(U) \longrightarrow \Gamma_c(X_y, \mathcal{F}_y).$$

Injectivity: Let $s \in (f_! \mathcal{F})(U)$ such that $s|_{X_y} = 0$. Thus $s \in \mathcal{F}(f^{-1}(U))$ and $\mathrm{supp}(s) \xrightarrow{f} U$ is proper. Since $s|_{X_y} = 0$, $f^{-1}(y) \cap \mathrm{supp}(s) = X_y \cap \mathrm{supp}(s) = \emptyset$, in particular $y \notin f(\mathrm{supp}(s))$. Let $y \in U'$ be the complement of $f(\mathrm{supp}(s))$ in U. Since $\mathrm{supp}(s) \xrightarrow{f} U$ is proper, $f(\mathrm{supp}(s))$ is closed in U, so U' is open in U and hence in Y. Moreover

$$f^{-1}(U') \cap \mathrm{supp}(s) \subseteq f^{-1}(U') \cap f^{-1}(f(\mathrm{supp}(s))) = f^{-1}(U' \cap f(\mathrm{supp}(s))) = f^{-1}(\emptyset) = \emptyset.$$

Hence $s|_{f^{-1}(U')} = 0$, so $s|_{U'} = 0$.

Surjectivity: Suppose first $\mathcal{F}$ is soft and let $s \in \Gamma_c(X_y, \mathcal{F}_y)$. Since $\mathcal{F}$ is soft, we may extend $s \in \mathcal{F}(X_y)$ to a compactly supported $s \in \mathcal{F}(X) = (f_*\mathcal{F})(Y)$. Since Y is Hausdorff, every compact $K \subseteq Y$ is closed and therefore its preimage under $f|_{\mathrm{supp}(s)}$ is closed in the compact $\mathrm{supp}(s)$, thus itself compact. Hence $f|_{\mathrm{supp}(s)} \colon \mathrm{supp}(s) \to Y$ is proper and $s \in (f_!\mathcal{F})(Y)$.

For arbitrary $\mathcal{F}$, there exists an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{J}$$

with $\mathcal{I}, \mathcal{J}$ soft (e.g. injective). The functors $(f_!\cdot)_y$ and $\Gamma_c(X_y, \cdot|_{X_y})$ are left exact, so we have a commuting diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (f_!\mathcal{F})_y & \longrightarrow & (f_!\mathcal{I})_y & \longrightarrow & (f_!\mathcal{J})_y \\ & & \downarrow & & \downarrow\simeq & & \downarrow\simeq \\ 0 & \longrightarrow & \Gamma_c(X_y, \mathcal{F}_y) & \longrightarrow & \Gamma_c(X_y, \mathcal{I}_y) & \longrightarrow & \Gamma_c(X_y, \mathcal{J}_y) \end{array} .$$

The five-lemma yields the desired isomorphism. □

Theorem 1.20. *Consider a cartesian diagram of locally compact Hausdorff spaces:*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{f} & Y \end{array} .$$

Then there is a natural isomorphism, for any $\mathcal{F}^\bullet \in \mathcal{D}^+(X)$,

$$f^*\mathrm{R}p_!\mathcal{F}^\bullet \simeq \mathrm{R}p'_!f'^*\mathcal{F}^\bullet.$$

Proof. By the universal property of derived functors, it suffices to define a natural transformation $f^*p_! \to \mathrm{R}p'_!f'^*$. By composing with the canonical natural transformation $p'_!f'^* \to \mathrm{R}p'_!f'^*$, it suffices to define the dotted arrow in the diagram below

$$\begin{array}{ccc} f^*p_! & \dashrightarrow & \mathrm{R}p'_!f'^* \\ & \searrow & \nearrow{\scriptstyle can} \\ & p'_!f'^* & \end{array} .$$

By naturality, it is sufficient to define for $\mathcal{G} \in \mathcal{A}b(X)$ a natural map $f^*p_!\mathcal{G} \to p'_!f'^*\mathcal{G}$. Since $f^* \dashv f_*$, this is equivalent to defining a natural map $p_!\mathcal{G} \to f_*p'_!f'^*\mathcal{G}$.

Again using $f'^* \dashv f'_*$, the map $\mathrm{id}_{f'^*\mathcal{G}}$ induces a map $\mathcal{G} \to f'_*f'^*\mathcal{G}$. Applying p_* yields $p_*\mathcal{G} \to p_*f'_*f'^*\mathcal{G}$. By the commutativity of the diagram we have $p_*f'_* = (pf')_* = (fp')_* = f_*p'_*$, so a map $\varphi \colon p_*\mathcal{G} \to f_*p'_*f'^*\mathcal{G}$.

For $U \subseteq Y$ open, this induces a map

$$\varphi_U \colon \mathcal{G}(p^{-1}(U)) \longrightarrow (f'^*\mathcal{G})(p'^{-1}(f^{-1}(U))).$$

Let now $s \in \mathcal{G}(p^{-1}(U))$ such that $\mathrm{supp}(s) \xrightarrow{p} U$ is proper. Since f'^* preserves stalks, for $(x,z) \in p^{-1}(U) \times_U f^{-1}(U)$ we have the following equivalences

$$(x,z) \in \mathrm{supp}(\varphi_U(s)) \iff \varphi_U(s)_{(x,z)} \neq 0 \iff s_{f'(x,z)} \neq 0 \iff s_x \neq 0 \iff x \in \mathrm{supp}(s).$$

Thus $\mathrm{supp}(\varphi_U(s)) = \mathrm{supp}(s) \times_U f^{-1}(U)$. We therefore have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{supp}(s) \times_U f^{-1}(U) & \longrightarrow & \mathrm{supp}(s) \\ \downarrow & & \downarrow \\ f^{-1}(U) & \longrightarrow & U \end{array} .$$

By assumption the right vertical arrow is proper. Since properness is stable under (topological) base change, the left vertical arrow is proper too. Hence $\operatorname{supp}(\varphi_U(s)) \xrightarrow{p'} f^{-1}(U)$ is proper and

$$\varphi_U(s) \in (p'_! f'^* \mathcal{G})(f^{-1}(U)) = (f_* p'_! f'^* \mathcal{G})(U).$$

Thus φ restricts to a natural map

$$p_! \mathcal{G} \longrightarrow f_* p'_! f'^* \mathcal{G}.$$

To check that this is an isomorphism, we can use the fact that both functors are way-out functors in the sense of Section 7 in [Har66]. Thus we only need to check this for a single sheaf $\mathcal{F} \in \mathcal{A}b(X)$, i.e. we want to show

$$f^* R^i p_! \mathcal{F} \xrightarrow{\simeq} R^i p'_! f'^* \mathcal{F}$$

for all $i \geq 0$. Again by universality of the δ-functors involved, we may assume $i = 0$. Moreover, we can check this at the level of stalks. Let $z \in Z$. Then on the left hand side

$$(f^* p_! \mathcal{F})_z \simeq (p_! \mathcal{F})_{f(z)} \overset{1.19}{\simeq} \Gamma_c(p^{-1}(f(z)), \mathcal{F}|_{p^{-1}(f(z))}) = \Gamma_c(f'(p'^{-1}(z))), \mathcal{F}|_{f'(p'^{-1}(z))}) \tag{1}$$

On the right hand side, we have

$$(p'_! f'^* \mathcal{F})_z \overset{1.19}{\simeq} \Gamma_c(p'^{-1}(z), (f'^* \mathcal{F})|_{p'^{-1}(z)}) \tag{2}$$

$\mathcal{F}|_{f'(p'^{-1}(z))}$ and $(f'^* \mathcal{F})|_{p'^{-1}(z)}$ are given as the sheafification of the same presheaf, indeed:

$$\begin{aligned} \operatorname*{colim}_{p'^{-1}(z) \subseteq U \subseteq X \times_Y Z} (f'^* \mathcal{F})(U) &= \operatorname*{colim}_{p'^{-1}(z) \subseteq U \subseteq X \times_Y Z} \; \operatorname*{colim}_{f'(U) \subseteq V \subseteq X} \mathcal{F}(V) \\ &= \operatorname*{colim}_{f'(p'^{-1}(z)) \subseteq V \subseteq X} \mathcal{F}(V). \end{aligned}$$

This shows (1) $\simeq$ (2) and concludes the proof. ☐

Proposition 1.21. *Soft sheaves are $f_!$-acyclic. In particular, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence in $\mathcal{A}b(X)$ and $\mathcal{F}'$ is soft, then the sequence $0 \to f_! \mathcal{F}' \to f_! \mathcal{F} \to f_! \mathcal{F}'' \to 0$ is exact.*

Proof. Let $i > 0$ and $\mathcal{F} \in \mathcal{A}b(X)$ be soft. Then for $y \in Y$

$$(R^i f_! \mathcal{F})_y \overset{1.19}{\simeq} H^i_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}) \overset{1.18}{=} 0,$$

since the restriction of a soft sheaf to a closed subset is soft. ☐

Example 1.22. Let $U \subseteq X$ be open and $j \colon U \to X$ the inclusion map. By looking at stalks, one finds that $j_! \mathcal{F}$ for $\mathcal{F} \in \mathcal{A}b(U)$ is just extension by zero.

Proposition 1.23 (Lower shriek preserves softness)**.** *If $f \colon X \to Y$ is continuous and $\mathcal{F} \in \mathcal{A}b(X)$ is soft, then $f_! \mathcal{F}$ is soft too.*

Proof. Let $Z \subseteq Y$ be compact and $s \in (f_! \mathcal{F})(Z) \simeq \operatorname*{colim}_{Z \subseteq U \subseteq Y} (f_! \mathcal{F})(U)$. Then there exists an open neighbourhood U of Z and an extension $\tilde{s} \in (f_! \mathcal{F})(U) \subseteq \mathcal{F}(f^{-1}(U))$ with $\operatorname{supp}(\tilde{s}) \xrightarrow{f} U$ proper. Since Y is locally compact, there exists a compact neighbourhood $L \subseteq U$ of Z. Restricting $\tilde{s}$ to the compact $K := \left(f|_{\operatorname{supp}(\tilde{s})}\right)^{-1}(L) \subseteq \operatorname{supp}(\tilde{s})$ and extending by softness of $\mathcal{F}$, yields a compactly supported global section $t \in \mathcal{F}(X) = (f_* \mathcal{F})(Y)$ such that $t|_Z = s$. Since $\operatorname{supp}(t)$ is compact and Y is Hausdorff, $\operatorname{supp}(t) \xrightarrow{f} Y$ is proper. ☐

Corollary 1.24 (Leray spectral sequence)**.** *Given maps $f \colon X \to Y$, $g \colon Y \to Z$ of locally compact Hausdorff spaces, there is a natural isomorphism* $\mathrm{R}(g \circ f)_! \simeq \mathrm{R}g_! \circ \mathrm{R}f_!$.

Proof. Since soft sheaves are $f_!$ (and $g_!$) acyclic and $f_!$ maps soft sheaves to soft sheaves, the result follows from Proposition 5.4 in [Har66]. ☐

References

[GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

[Har66] Robin Hartshorne. *Residues and duality*. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin-New York, 1966. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, With an appendix by P. Deligne.

[KS94] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1994. With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original.

[Mat] Akhil Mathew. Verdier duality. Expository Notes (version dated July 29, 2011), available at `https://math.uchicago.edu/~amathew/verd.pdf`.