1. Preliminaries

These notes mostly follow [Mat]. Some ideas are taken from [GM03] and [KS94].

In the following, a topological space X is always assumed to be locally compact and Hausdorff. Denote by $\mathcal{A}b(X)$ the category of sheaves of abelian groups on X. Furthermore, denote by $\mathrm{D}(X)$ (respectively $\mathrm{D}^+(X)$) the (bounded below) derived category of $\mathcal{A}b(X)$.

Definition 1.1 (Lower Shriek)**.** Let $f\colon X \to Y$ be a continuous map of spaces. For $\mathcal{F} \in \mathcal{A}b(X)$ and $U \subseteq Y$ open, let

$$f_!(\mathcal{F})(U) = \{s \in \mathcal{F}(f^{-1}(U))\colon \operatorname{supp}(s) \xrightarrow{f} U \text{ proper}\}.$$

Remark 1.2 (Support)**.** For $\mathcal{F} \in \mathcal{A}b(X)$, $U \subseteq X$ open and a section $s \in \mathcal{F}(U)$, its support $\operatorname{supp}(s)$ is defined as

$$\{x \in U\colon s_x \neq 0\}.$$

This set is always closed, as its complement is open.

Lemma 1.3 (Lower shriek of sheaf is a sheaf)**.** *Let $\mathcal{F} \in \mathcal{A}b(X)$ be a sheaf $f\colon X \to Y$ continuous. Then $f_!\mathcal{F}$ is a sheaf on Y.*

Proof. Clearly, $f_!\mathcal{F}$ is a sub-presheaf of the sheaf $f_*\mathcal{F}$. To show it is a sheaf, we need to verify that gluing sections in $f_!\mathcal{F}$ gives again a section in $f_!\mathcal{F}$.

Let $(U_i)_{i\in I}$ be a family of open sets in Y and $s_i \in (f_!\mathcal{F})(U_i)$ sections. Thus $s_i \in \mathcal{F}(f^{-1}(U_i))$ such that $\operatorname{supp}(s_i) \xrightarrow{f} U_i$ is proper. Gluing yields a unique section $s \in \mathcal{F}(f^{-1}(U))$. We need to check that

$$\operatorname{supp}(s) = \bigcup_{i\in I} \operatorname{supp}(s_i) \xrightarrow{f} \bigcup_{i\in I} U_i$$

is proper. For this note that $\left(f|_{\operatorname{supp}(s)}\right)^{-1}(U_i) = f^{-1}(U_i) \cap \operatorname{supp}(s) = \operatorname{supp}(s_i)$ and being proper is local on the target. □

Remark 1.4 (Lower shriek is left exact)**.** Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ be an exact sequence in $\mathcal{A}b(X)$ and $f\colon X \to Y$ continuous. Then

$$0 \to f_!\mathcal{F}' \to f_!\mathcal{F} \to f_!\mathcal{F}''$$

is exact.

Proof. We have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & f_!\mathcal{F}' & \longrightarrow & f_!\mathcal{F} & \longrightarrow & f_!\mathcal{F}'' \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & f_*\mathcal{F}' & \longrightarrow & f_*\mathcal{F} & \longrightarrow & f_*\mathcal{F}'' \end{array},$$

where the second row is exact. Thus the claim follows. □

Remark 1.5 (Lower shriek and compact support)**.** Let $f\colon X \to \{*\}$ be the unique continuous map from X to the one point space and $\mathcal{F} \in \mathcal{A}b(X)$. Then

$$(f_!\mathcal{F})(\{*\}) = \{s \in \mathcal{F}(X)\colon \operatorname{supp}(s) \to \{*\} \text{ proper}\} = \{s \in \mathcal{F}(X)\colon \operatorname{supp}(s) \text{ compact}\}.$$

Denote this by $\Gamma_c(X, \mathcal{F})$.

2. Derivative of lower shriek

The goal of this and the following talk is to prove the following theorem

Theorem 2.1 (Verdier duality). *If X, Y are locally compact topological spaces of finite dimension, then $\mathrm{R}f_!$ admits a right adjoint $f^!\colon \mathrm{D}^+(Y) \to \mathrm{D}(X)$.*

To calculate the derivative of $f_!$, we need to introduce an adapted class of sheaves.

Definition 2.2. Let X be space, $\mathcal{F} \in \mathcal{A}b(X)$ and $Z \subseteq X$ a subset. Then define

$$\mathcal{F}(Z) = \Gamma(Z, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$$

where $\mathcal{F}|_Z = i^*\mathcal{F}$ for $i\colon Z \to X$ the canonical inclusion.

Remark 2.3 (Lemma 1.4 in [Mus]). If $\mathcal{F} \in \mathcal{A}b(X)$, $Z_1, Z_2 \subseteq X$ are closed and $t_1 \in \mathcal{F}(Z_1)$, $t_2 \in \mathcal{F}(Z_2)$ are given such that $t_1|_{Z_1\cap Z_2} = t_2|_{Z_1\cap Z_2}$, then there exists a unique section $t \in \mathcal{F}(Z_1 \cup Z_2)$ such that $t|_{Z_1} = t_1$ and $t|_{Z_2} = t_2$.

Remark 2.4. If $Z \subseteq X$ is a subset and $i\colon Z \to X$ the canonical inclusion, then

$$\mathcal{F}(Z) = \left\{ (s_i, U_i)_{i\in I}\colon U_i \subseteq X \text{ open with } Z \subseteq \bigcup_{i\in I} U_i, s_i \in \mathcal{F}(U_i) \text{ with } (s_i)_z = (s_j)_z \forall i, j \in I, z \in Z \cap U_i \cap U_j \right\} / \sim .$$

where $(U_i, s_i)_{i\in I} \sim (V_j, t_j)_{j\in J}$ if and only if $(s_i)_z = (t_j)_z$ for all $i \in I$, $j \in J$ and $z \in U_i \cap V_j \cap Z$.

For every open neighbourhood U of Z, we have a restriction map

$$\mathcal{F}(U) \to \mathcal{F}(Z), s \mapsto s|_Z := [(s, U)].$$

This induces a map

$$\operatorname*{colim}_{Z\subseteq U} \mathcal{F}(U) \to \mathcal{F}(Z).$$

Lemma 2.5. *Let X be a space and $\mathcal{F} \in \mathcal{A}b(X)$. If $Z \subseteq X$ is compact, the natural map*

$$\operatorname*{colim}_{Z\subseteq U} \mathcal{F}(U) \longrightarrow \mathcal{F}(Z)$$

is an isomorphism.

Proof. Injectivity: Let $s \in \mathcal{F}(U)$ such that $s|_Z = 0$. Thus for all $z \in Z$, $s_z = 0$ and there exists an open neighbourhood $z \in U_z \subseteq U$ such that $s|_{U_z} = 0$. Thus $s|_{\bigcup U_z} = 0$. Since $Z \subseteq \bigcup_{z\in Z} U_z$, s is zero in the colimit.

Surjectivity: Take $(s_i, U_i)_{i\in I} \in \mathcal{F}(Z)$. Thus $Z \subseteq \bigcup_{i\in I} U_i$ and by local compactness, for every $z \in Z$, there exists a compact neighbourhood $z \in K_z$ such that $K_z \subseteq U_{i_z}$ for some $i_z \in I$. Since Z is compact, finitely many suffice, so we may assume $Z \subseteq \bigcup_{i=1}^n K_i$ and $K_i \subseteq U_i \subseteq X$. We now want to define a section on a neighbourhood of Z that locally agrees with the s_i.

By induction, we may assume $n = 2$. By definition, $(s_1)_z = (s_2)_z$ for all $z \in Z \cap U_1 \cap U_2$, in particular $s_1|_{U_1\cap U_2}$ and $s_2|_{U_1\cap U_2}$ have the same restriction to $K_1 \cap K_2$. By the injectivity of the restriction map, there exists an open neighbourhood $K_1 \cap K_2 \subseteq V \subseteq U_1 \cap U_2$, such that $s_1|_V = s_2|_V$. Since $K_j \setminus V$ is closed in the compact K_j, for $j = 1, 2$ the subset $K_j \setminus V$ is compact. Since X is Hausdorff, there exist open neighbourhoods $K_j \setminus V \subseteq U_j' \subseteq U_j$ such that $U_1' \cap U_2' = \emptyset$. Now $s_1|_{U_1'}$, $s_2|_{U_2'}$ and $s_1|_V = s_2|_V$ glue to a section w on $U_1' \cup U_2' \cup V \supseteq K_1 \cup K_2 \supseteq Z$ such that $w|_Z = [(s_i, U_i)_{i\in I}]$. □

Definition 2.6. A sheaf $\mathcal{F} \in \mathcal{A}b(X)$ is ***soft*** if $\mathcal{F}(X) \to \mathcal{F}(Z)$ is surjective whenever $Z \subseteq X$ is compact.

Remark 2.7. In [KS94] our notion of softness is called *c-soft*. For σ-compact spaces the notions agree according to Exercise II.6 in [KS94].

Remark 2.8 (Flasque sheaves are soft)**.** Recall that a sheaf $\mathcal{F} \in \mathcal{A}b(X)$ is called *flasque*, if for every open set $U \subseteq X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. For $Z \subseteq X$ compact, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\qquad\qquad\qquad} & \mathcal{F}(Z) \\ & \searrow\!\!\!\!\searrow \qquad \nearrow_{\simeq} & \\ & \operatorname*{colim}_{Z \subseteq U} \mathcal{F}(U) & \end{array} .$$

Thus $\mathcal{F}$ is soft.

Remark 2.9 (Prop. 2.5.6 in [KS94])**.** Let $\mathcal{F} \in \mathcal{A}b(X)$. Then $\mathcal{F}$ is soft if and only if for any closed subset $Z \subseteq X$, the restriction $\Gamma_c(X, \mathcal{F}) \to \Gamma_c(Z, \mathcal{F}|_Z)$ is surjective.

Proof. If $K \subseteq X$ is compact, $\Gamma(K, F) = \Gamma_c(K, F|_K)$, so the condition is sufficient. Conversely assume $\mathcal{F}$ is soft and let $s \in \Gamma_c(Z, \mathcal{F}|_Z)$ with compact support K. Let U be a relatively compact open neighbourhood of K in X. Define $\tilde{s} \in \Gamma(\partial U \cup (Z \cap \overline{U}), \mathcal{F})$ by setting $\tilde{s}_{Z \cap \overline{U}} = s$ and $\tilde{s}|_{\partial U} = 0$. By softness, this extends to a global section $t \in \Gamma(X, \mathcal{F})$. Since $t = 0$ on a neighbourhood of ∂U, we may assume t is supported by $\overline{U}$. □

Example 2.10. Let M be a smooth manifold and let $f \in \mathcal{C}^\infty(K)$ be a section over a compact set K, i.e. a smooth function defined on some neighbourhood U of K. Thus by using a partition of unity, we can extend f to a global smooth function $\tilde{f} \in \mathcal{C}^\infty(M)$ such that $\tilde{f}|_K = f$. In other words, the sheaf $\mathcal{C}^\infty$ is soft.

In a similar fashion we see that the sheaf of sections of a smooth vector bundle on M is soft.

Example 2.11. If $\mathcal{A}$ is a soft sheaf of rings and $\mathcal{F}$ is a sheaf of $\mathcal{A}$-modules, then $\mathcal{F}$ is soft. Indeed, let $s \in \mathcal{F}(K)$ be a section over a compact set $K \subseteq X$, i.e. a section on some open neighbourhood of K. By softness we can extend the section $1 \in \mathcal{A}(K)$ to a compactly supported global section $i \in \mathcal{A}(X)$ with support in U. Thus si extends to a global section of $\mathcal{F}$.

Proposition 2.12. *Let X be a space. If $\mathcal{F} \in \mathcal{A}b(X)$ is soft, $K \subseteq X$ is compact and $K \subseteq U$ is an open neighbourhood, any section over K can be extended to a global section with compact support contained in U.*

Proof. Let $s \in \mathcal{F}(K)$. By local compactness, there exists a compact neighbourhood L of K with $L \subseteq U$. Then $K \cap \partial L = \emptyset$. Consider the section on $K \cup \partial L$ given by s on K and zero on ∂L. Since $\mathcal{F}$ is soft, this can be extended to a global section, and a fortiori to a section t over L. Now the sections given by t on L and 0 on $\overline{X \setminus L}$ glue to a compactly supported extension of s. Since $L \subseteq U$, its support is contained in U. □

2.1. **Compactly supported cohomology.** Let X be a space.

Theorem 2.13 (Base change)**.** *Let $f \colon X \to Y$ be a continuous map of spaces. For $\mathcal{F} \in \mathcal{A}b(X)$, there is a natural isomorphism*

$$(f_!\mathcal{F})_y \simeq \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$$

for each $y \in Y$.

Proof. Denote by X_y the fibre of f over y and by $\mathcal{F}$ the restriction to X_y. Let $y \in U \subseteq Y$ open. Then consider the natural map

$$\begin{aligned} (f_!\mathcal{F})(U) &\longrightarrow \Gamma_c(X_y, \mathcal{F}_y) \\ s &\longmapsto s|_{X_y}. \end{aligned}$$

This is well-defined, since for any $s \in \mathcal{F}(f^{-1}(U))$ with $\operatorname{supp}(s) \xrightarrow{f} U$ proper, we have

$$\operatorname{supp}(s|_{X_y}) = \operatorname{supp}(s) \cap X_y = \left(f|^U_{\operatorname{supp}(s)}\right)^{-1}(y)$$

and the right hand side is compact. This map induces a natural map

$$(f_!\mathcal{F})_y = \operatorname*{colim}_{y\in U\subseteq Y} (f_!\mathcal{F})(U) \longrightarrow \Gamma_c(X_y, \mathcal{F}_y).$$

Injectivity: Let $s \in (f_!\mathcal{F})(U)$ such that $s|_{X_y} = 0$. Thus $s \in \mathcal{F}(f^{-1}(U))$ and $\operatorname{supp}(s) \xrightarrow{f} U$ is proper. Since $s|_{X_y} = 0$, $f^{-1}(y) \cap \operatorname{supp}(s) = X_y \cap \operatorname{supp}(s) = \emptyset$, in particular $y \notin f(\operatorname{supp}(s))$. Let $y \in U'$ be the complement of $f(\operatorname{supp}(s))$ in U. Since $\operatorname{supp}(s) \xrightarrow{f} U$ is proper, $f(\operatorname{supp}(s))$ is closed in U, so U' is open in U and hence in Y. Moreover

$$f^{-1}(U') \cap \operatorname{supp}(s) \subseteq f^{-1}(U') \cap f^{-1}(f(\operatorname{supp}(s))) = f^{-1}(U' \cap f(\operatorname{supp}(s))) = f^{-1}(\emptyset) = \emptyset.$$

Hence $s|_{f^{-1}(U')} = 0$, so $s|_{U'} = 0$.

Surjectivity: Suppose first $\mathcal{F}$ is soft and let $s \in \Gamma_c(X_y, \mathcal{F}_y)$. Since $\mathcal{F}$ is soft, we may extend $s \in \mathcal{F}(X_y)$ to a compactly supported $s \in \mathcal{F}(X) = (f_*\mathcal{F})(Y)$. Since Y is Hausdorff, every compact $K \subseteq Y$ is closed and therefore its preimage under $f|_{\operatorname{supp}(s)}$ is closed in the compact $\operatorname{supp}(s)$, thus itself compact. Hence $f|_{\operatorname{supp}(s)}\colon \operatorname{supp}(s) \to Y$ is proper and $s \in (f_!\mathcal{F})(Y)$.

For arbitrary $\mathcal{F}$, there exists an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{J}$$

with $\mathcal{I}, \mathcal{J}$ soft (e.g. injective). The functors $(f_!\cdot)_y$ and $\Gamma_c(X_y, \cdot|_{X_y})$ are left exact, so we have a commuting diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & (f_!\mathcal{F})_y & \longrightarrow & (f_!\mathcal{I})_y & \longrightarrow & (f_!\mathcal{J})_y & \\
& \downarrow & & \downarrow\simeq & & \downarrow\simeq & . \\
0 \longrightarrow & \Gamma_c(X_y, \mathcal{F}_y) & \longrightarrow & \Gamma_c(X_y, \mathcal{I}_y) & \longrightarrow & \Gamma_c(X_y, \mathcal{J}_y) &
\end{array}$$

The five-lemma yields the desired isomorphism. □

Proposition 2.14 (Lower shriek is exact on soft)**.** *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence in $\mathcal{A}b(X)$ with $\mathcal{F}'$ soft. Then the sequence*

$$0 \to f_!\mathcal{F}' \to f_!\mathcal{F} \to f_!\mathcal{F}'' \to 0$$

is exact.

Proof. Since $f_!$ is left exact, we only need to show the surjectivity on the right, i.e. for every $y \in Y$ the surjectivity of $(f_!\mathcal{F})_y \to (f_!\mathcal{F}'')_y$. We have the following commutative diagram:

$$\begin{array}{ccc}
\Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}) & \longrightarrow & \Gamma_c(f^{-1}(y), \mathcal{F}''|_{f^{-1}(y)}) \\
\downarrow & & \downarrow \quad . \\
(f_!\mathcal{F})_y & \longrightarrow & (f_!\mathcal{F}'')_y
\end{array}$$

By 2.13, the vertical arrows are isomorphisms. It suffices thus to show the surjectivity of $\Gamma_c(f^{-1}(y), \mathcal{F}_{f^{-1}(y)}) \to \Gamma_c(f^{-1}(y), \mathcal{F}''|_{f^{-1}(y)})$. Restriction to $f^{-1}(y)$ is exact, moreover it preserves softness. We thus reduced to showing that $\Gamma_c(X, \cdot)$ preserves surjections.

Suppose first that X is compact and let $s \in \Gamma_c(X, \mathcal{F}'') = \Gamma(X, \mathcal{F}'')$. Since $\mathcal{F} \to \mathcal{F}'' \to 0$ is exact, there exist a covering $X = \bigcup_{i\in I} U_i$ and lifts $t_i \in \mathcal{F}(U_i)$ of $s|_{U_i}$. By local compactness of X, we may assume, after a possible refinement, that each U_i contains a compact set V_i whose interiors still cover X. Since X is compact, we may assume I is finite. To piece together the t_i, we may assume, by induction, that $\#I = 2$.

Consider $t_1|_{U_1\cap U_2} - t_2|_{U_1\cap U_2}$. This is necessarily a section e' of $\mathcal{F}'(U_1 \cap U_2)$ as it maps to zero in $\mathcal{F}''(U_1 \cap U_2)$. Restricting e' to the compact $V_1 \cap V_2$ and extending it by softness, yields a global section e of $\mathcal{F}'$. Now

$$(t_2|_{V_2} + e|_{V_2})|_{V_1\cap V_2} = t_2|_{V_1\cap V_2} + e'|_{V_1\cap V_2} = t_1|_{V_1\cap V_2}.$$

Thus $t_1|_{V_1}, t_2|_{V_2} + e|_{V_2}$ glue to a global section t of $\mathcal{F}$ with image s.

Now for general X: Let $s \in \mathcal{F}''(X)$ with compact support Z. By local compactness, there exists a compact neighbourhood $Z' \subseteq X$ of Z. Since pullback of sheaves is exact and restriction of soft sheaves to closed subsets preserves softness, applying the result to Z', yields a section $t' \in \mathcal{F}(Z')$ lifting $s|_{Z'}$. The restriction $t'|_{\partial Z'}$ maps to $s|_{\partial Z'} = 0$, so $t'|_{\partial Z'} \in \mathcal{F}'(\partial Z')$. Since $\partial Z'$ is compact and $\mathcal{F}'$ is soft, $t'|_{\partial Z'}$ extends to a global section b of $\mathcal{F}'$. Thus

$$(t' - b|_{Z'})|_{\partial Z'} = t'|_{\partial Z'} - t'|_{\partial Z'} = 0.$$

So $t' - b|_{Z'}$ on Z' and 0 on $\overline{X \setminus Z'}$ glue to a global section t of $\mathcal{F}$. Then $t|_{Z'} = t' - b|_{Z'}$ maps to $s|_{Z'}$ since $b \in \mathcal{F}'(X)$. Since $\operatorname{supp}(t), \operatorname{supp}(s) \subseteq Z'$, t is a compactly supported lift of s. □

Corollary 2.15. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence in $\mathcal{A}b(X)$ and $\mathcal{F}', \mathcal{F}$ are soft, then $\mathcal{F}''$ is soft too.*

Proof. Let $Z \subseteq X$ be closed. Since restricting to a closed subset is exact and preserves softness, by 2.14 $\Gamma_c(Z, \mathcal{F}) \to \Gamma_c(Z, \mathcal{F}'')$ is surjective. This yields a commutative diagram

$$\begin{array}{ccc} \Gamma_c(X,\mathcal{F}) & \longrightarrow & \Gamma_c(X,\mathcal{F}'') \\ \downarrow\!\!\downarrow & & \downarrow \\ \Gamma_c(Z,\mathcal{F}) & \twoheadrightarrow & \Gamma_c(Z,\mathcal{F}'') \end{array},$$

where the left vertical arrow is surjective, since $\mathcal{F}$ is soft. Since the composition is surjective, $\Gamma_c(X, \mathcal{F}'') \to \Gamma_c(Z, \mathcal{F}'')$ is also surjective. □

3. Derived categories and functors

We give a brief introduction to the derived category of an abelian category $\mathcal{A}$. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor and let $\mathcal{A}$ have enough injectives. Then the classical derived functors exist. To compute $\mathrm{R}^i F(X)$ for an object $X \in \mathcal{A}$, we choose an injective resolution

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_0 & \longrightarrow & I_1 & \longrightarrow & I_2 & \longrightarrow & \cdots \end{array}$$

i.e. a quasiisomorphism $X \to I^\bullet$. Then $\mathrm{R}^i F(X) = H^i F(I^\bullet)$.

New idea: identify X with its resolution, in other words, turn quasiisomorphisms into isomorphisms. First step in this direction: Consider the category $\mathcal{K}(\mathcal{A})$ of complexes where arrows are homomorphisms of complexes up to homotopy. Still quasiisomorphisms are in general not isomorphisms, so need to do more:

Localise by the class of quasiisomorphisms. This is then called the derived category of $\mathcal{A}$:

$$\mathcal{D}(\mathcal{A}) = \mathcal{K}(\mathcal{A})_{\mathcal{Q}is}.$$

Exactly like in the situation for rings, not every functor $\mathcal{K}(A) \to \mathcal{K}(B)$ descends to the derived category, since it needs to send quasiisomorphisms to quasiisomorphisms. If the functor is induced by an exact functor $\mathcal{A} \to \mathcal{B}$, this is the case. For an arbitrary $F\colon \mathcal{A} \to \mathcal{B}$, we can hope that a derived functor exists. This is defined by a universal property, that ensures that this derived functor is in a sense close to the original one.

For a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$, there is the following result:

Theorem 3.1. *If there exists a full additive subcategory $\mathcal{L}$ in $\mathcal{A}$ that is* adapted *to F, i.e.*

(i) for any $X \in \mathcal{A}$ there exists $X' \in \mathcal{L}$ and an exact sequence $0 \to X \to X'$
(ii) if $0 \to X' \to X \to X'' \to 0$ is exact sequence in $\mathcal{A}$ and X', X are in $\mathcal{L}$, then X'' is in $\mathcal{L}$
(iii) if $0 \to X' \to X \to X'' \to 0$ is exact sequence in $\mathcal{A}$ and if X', X, X'' are in $\mathcal{L}$, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

Then the derived functor $\mathrm{R}F\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ exists and for any $I^\bullet \in \mathcal{K}^+(\mathcal{L})$ we have a natural isomorphism

$$\mathrm{R}F(I^\bullet) \simeq F(I^\bullet).$$

Since $\mathcal{A}b(X)$ has enough injectives and every injective sheaf is soft, by 2.14 and 2.15, the class of soft sheaves is adapted to the functor $f_!$. Thus the derived functor

$$\mathrm{R}f_!\colon \mathcal{D}^+(X) \longrightarrow \mathcal{D}^+(Y)$$

exists.

Corollary 3.2. *For $\mathcal{F}^\bullet \in \mathcal{K}om^+(\mathcal{A}b(X))$, we have a natural isomorphism*

$$(\mathrm{R}f_!\mathcal{F}^\bullet)_y \simeq \mathrm{R}\Gamma_c(f^{-1}(y), \mathcal{F}^\bullet|_{f^{-1}(y)})$$

in $\mathcal{D}(X)$.

Proof. Let $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$\begin{aligned}(\mathrm{R}f_!\mathcal{F}^\bullet)_y &\simeq (\mathrm{R}f_!\mathcal{I}^\bullet)_y\\ &\simeq (f_!\mathcal{I}^\bullet)_y\\ &\simeq \Gamma_c(f^{-1}(y), \mathcal{I}^\bullet|_{f^{-1}(y)})\\ &\simeq \mathrm{R}\Gamma_c(f^{-1}(y), \mathcal{I}^\bullet|_{f^{-1}(y)})\\ &\simeq \mathrm{R}\Gamma_c(f^{-1}(y), \mathcal{F}^\bullet|_{f^{-1}(y)}).\end{aligned}$$

□

Example 3.3. Let $U \subseteq X$ be open and $j\colon U \to X$ the inclusion map. By looking at stalks, one finds that $j_!\mathcal{F}$ for $\mathcal{F} \in \mathcal{A}b(U)$ is just extension by zero.

Proposition 3.4 (Lower shriek preserves softness)**.** *If $f\colon X \to Y$ is continuous and $\mathcal{F} \in \mathcal{A}b(X)$ is soft, then $f_!\mathcal{F}$ is soft too.*

Proof. Let $Z \subseteq Y$ be compact and $s \in (f_!\mathcal{F})(Z) \simeq \operatorname{colim}_{Z\subseteq U\subseteq Y} (f_!\mathcal{F})(U)$. Then there exists an open neighbourhood U of Z and an extension $\tilde{s} \in (f_!\mathcal{F})(U) \subseteq \mathcal{F}(f^{-1}(U))$ with $\operatorname{supp}(\tilde{s}) \xrightarrow{f} U$ proper. Since Y is locally compact, there exists a compact neighbourhood $L \subseteq U$ of Z. Restricting $\tilde{s}$ to the compact $K := \left(f|_{\operatorname{supp}(\tilde{s})}\right)^{-1}(L) \subseteq \operatorname{supp}(\tilde{s})$ and extending by softness of $\mathcal{F}$, yields a compactly supported global section $t \in \mathcal{F}(X) = (f_*\mathcal{F})(Y)$ such that $t|_Z = s$. Since $\operatorname{supp}(t)$ is compact and Y is Hausdorff, $\operatorname{supp}(t) \xrightarrow{f} Y$ is proper. □

Corollary 3.5 (Leray spectral sequence)**.** *Given continuous maps $f\colon X \to Y$, $g\colon Y \to Z$ of spaces, there is a natural isomorphism* $\mathrm{R}(g \circ f)_! \simeq \mathrm{R}g_! \circ \mathrm{R}f_!$.

Proof. Since soft sheaves are $f_!$ (and $g_!$) acyclic and $f_!$ maps soft sheaves to soft sheaves, the result follows from Proposition 5.4 in [Har66]. □

4. Other functors on abelian sheaves

Functor	Exactness	Derivative	Adapted class
$f^*\colon \mathcal{A}b(Y) \to \mathcal{A}b(X)$	exact	$f^*\colon \mathcal{D}(Y) \to \mathcal{D}(X)$	
$f_*\colon \mathcal{A}b(X) \to \mathcal{A}b(Y)$	left exact	$\mathrm{R}f_*\colon \mathcal{D}(X) \to \mathcal{D}(Y)$	K-limp complexes
$\cdot \otimes \mathcal{F}\colon \mathcal{A}b(X) \to \mathcal{A}b(X)$	right exact	$\cdot \otimes^L \mathcal{F}^\bullet\colon \mathcal{D}(X) \to \mathcal{D}(X)$	K-flat complexes
$\underline{\mathrm{Hom}}(\mathcal{F}, \cdot)\colon \mathcal{A}b(X) \to \mathcal{A}b(X)$	left exact	$\mathrm{R}\underline{\mathrm{Hom}}^\bullet(\mathcal{F}^\bullet, \cdot)\colon \mathcal{D}(X) \to \mathcal{D}(X)$	K-injective complexes
$f_!\colon \mathcal{A}b(X) \to \mathcal{A}b(Y)$	left exact	$\mathrm{R}f_!\colon \mathcal{D}^+(X) \to \mathcal{D}^+(Y)$	soft sheaves
		$f^!\colon \mathcal{D}^+(Y) \to \mathcal{D}(X)$	

The internal Hom functor is for $\mathcal{F}, \mathcal{G} \in \mathcal{A}b(X)$ given by the formula

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{A}b(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$$

for every $U \subseteq X$ open and the (internal) tensor product by the sheafification of the presheaf

$$U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U).$$

These functors satisfy the following adjunction results

$$f^* \dashv \mathrm{R}f_*$$

and

$$\cdot \otimes^L \mathcal{F}^\bullet \dashv \mathrm{R}\underline{\mathrm{Hom}}(\mathcal{F}^\bullet, \cdot).$$

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