VL Woche vom 28. 4. und 30. 4.

Nochmal zur Wiederholung: $D \subseteq \mathbb{R}^n$ offen, $f: D \to \mathbb{R}^m$

f heißt (total) diff'bar in $a \in D$ $\overset{df}{\iff}$ $\exists$ lineare Abb.: $Df(a): \mathbb{R}^n \to \mathbb{R}^m$
und eine Abb. $\varphi: B_\delta(0) \subset \mathbb{R}^n \to \mathbb{R}^m$ mit
i) $f(a+h) = f(a) + Df(a) \cdot h + \varphi(h) \quad \forall h \in B_\delta(0)$
ii) $\varphi(0) = 0$ und $\lim\limits_{h \to 0} \frac{\varphi(h)}{|h|} = 0$

Eigenschaften: · f diff'bar in $a \in D$ $\Rightarrow$ f partiell diff'bar in a

und $$Df(a) = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}}_{J_f(a) \quad \text{„Jacobi-Matrix"}}$$

· f diff'bar in $a \in D$ $\Rightarrow$ alle Richtungsableitungen existieren in a!
mit $D_v f(a) = Df(a) \cdot v$

Wichtig:

$$\begin{tikzcd}
\text{stetig part. diff'bar} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \text{partiell Diff'bar} & \text{kein Zshg.} \\
\text{stetig diff'bar} \arrow[r, Rightarrow] & \text{diff'bar} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{stetig} \\
& \text{alle Richtungsabl. existieren.} &
\end{tikzcd}$$

Ableitungsregeln: wie für $n=1$:

$D(f \circ g)(a) = D f(g(a)) \circ D f(a)$

$D(f \cdot g)(a) = f(a)\, Dg(a) + g(a)\, Df(a)$

$D\left(\frac{1}{g}\right)(a) = -\frac{1}{g(a)^2} D g(a)$

Höhere Ableitungen: $f \in C^k(D) = C^k(D, \mathbb{R}) \iff f$ k-mal stetig partiell diff'bar

$f \in C^0(D \iff f$ stetig.

Satz vom Schwarz: $f \in C^2(D)$: $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$

Hesse Matrix: $f \in C^2(D)$: $H_f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}$

$\leadsto$ später noch relevant!

A1! a) $p: \mathbb{R}^2 \to \mathbb{R}, \ (x,y) \mapsto x^2 + xy + y^2.$

$$\frac{\partial p}{\partial x} = 2x+y \quad \frac{\partial p}{\partial y} = 2y+x \quad \Rightarrow D_a p = \left(\frac{\partial p}{\partial x}\Big|_a \quad \frac{\partial p}{\partial y}\Big|_a \right)$$

$$= (2x+y \quad 2y+x) \in \mathbb{R}^{1\times 2}$$

$$\text{grad}(p) = (D_a p)^t = \begin{pmatrix} 2x+y \\ 2y+x \end{pmatrix}$$

$$\text{div}(\text{grad}(p)) = \frac{\partial(2x+y)}{\partial x} + \frac{\partial(2y+x)}{\partial y} = 2+2 = 4$$

b) i) $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \ (x,y) \mapsto \langle x,y \rangle = \sum_{i=1}^{n} x_i y_i$

$\frac{\partial f}{\partial x_i} = y_i \quad \frac{\partial f}{\partial y_i} = x_i$ stetig, es gilt dann

$\text{grad} f = (y, x)$

Sei nun $v \in \mathbb{R}^n \times \mathbb{R}^n$ bel., dann ist

$$D_{(v_x, v_y)} f(x,y) = D_{(x,y)} f \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \langle y, v_x \rangle + \langle x, v_y \rangle$$

ii) $g: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \ x \mapsto \frac{1}{|x|} = \frac{1}{\sqrt{\sum_{i=1}^{n} x_i^2}} = \left(\sum_{i=1}^{n} x_i^2 \right)^{-1/2}$

$\frac{\partial g}{\partial x_i} = -\frac{1}{2} \cdot \frac{2x_i}{\left(\sum_{i=1}^{n} x_i^2\right)^{3/2}} = -\frac{x_i}{|x|^3}$ stetig, es gilt dann

$$\text{grad } g = -\left(\frac{x_1}{|x|^3} \quad \frac{x_2}{|x|^3} \quad \cdots \quad \frac{x_n}{|x|^3} \right)^t = -\frac{x}{|x|^3}$$

c) $f: \mathbb{R} \to \mathbb{R}, \ x \mapsto \log(x^2), \quad g: \mathbb{R}^2 \to \mathbb{R}, \ (x,y) \mapsto \cos(x) + \sin(y).$

$f \circ g: \mathbb{R}^2 \to \mathbb{R}, \ (x,y) \mapsto \log\left((\cos(x) + \sin(y))^2\right) \left(= 2 \log(\cos(x) + \sin(y))\right)$ alternative Möglichkeit

$$\frac{\partial (f\circ g)}{\partial x} = \frac{1}{(\cos(x) + \sin(y))^2} \cdot 2(\cos(x) + \sin(y)) \cdot (-\sin(x)) = -\frac{2\sin(x)}{\cos(x) + \sin(y)}$$

$$\frac{\partial (f\circ g)}{\partial y} = \frac{2\cos(y)}{\cos(x) + \sin(y)}$$

Also $D_{(x,y)}(f\circ g) = \left(-\frac{2\sin(x)}{\cos(x)+\sin(y)} \quad \frac{2\cos(y)}{\cos(x)+\sin(y)} \right)$

Nun mit Kettenregel: $D_x f = \frac{2x}{x^2} = \frac{2}{x}, \ D_{(x,y)} g = \left(\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y} \right) = \left(-\sin(x) \quad \cos(y) \right)$

$$\Rightarrow D_{(x,y)}(f\circ g) = D_{g(x,y)} f \cdot D_{(x,y)} g = \frac{2}{g(x,y)} \left(-\sin(x) \quad \cos(y) \right) = \frac{2}{\cos(x)+\sin(y)} \left(-\sin(x) \quad \cos(y) \right)$$

$$= \left(\frac{-2\sin(x)}{\cos(x)+\sin(y)} \quad \frac{2\cos(y)}{\cos(x)+\sin(y)} \right)$$

d) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}, \ (x,y) \longmapsto x^2+y^2, \quad g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ (x,y) \longmapsto (\cos(x)+\sin(y), \sin(x)+\cos(y))$

$$f\circ g: \mathbb{R}^2 \longrightarrow \mathbb{R}, \ (x,y) \longmapsto (\cos(x)+\sin(y))^2 + (\sin(x)+\cos(y))^2$$

$$= \cos(x)^2 + 2\cos(x)\sin(y) + \sin(y)^2$$

$$+ \sin(x)^2 + 2\sin(x)\cos(y) + \cos(y)^2$$

$$= 2 + 2\cos(x)\sin(y) + 2\sin(x)\cos(y)$$

$$= 2 + \sin(x+y) - \sin(x-y) + \sin(x+y) + \sin(x-y)$$

$$= 2 + 2 \sin(x+y)$$

$$\frac{\partial(f\circ g)}{\partial x} = 2\cos(x+y) \qquad \frac{\partial(f\circ g)}{\partial y} = 2\cos(x+y)$$

Nun mit Kettenregel: $D_{(x,y)} f = \begin{pmatrix} 2x & 2y \end{pmatrix}$

$$D_{(x,y)}\, g = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -\sin(x) & \cos(y) \\ \cos(x) & -\sin(y) \end{pmatrix}$$

Dann folgt $D_{(x,y)}(f\circ g) = D_{g(x,y)} f \cdot D_{(x,y)}\, g$

$$= \begin{pmatrix} 2\cos x + 2\sin y & 2\sin x + 2\cos y \end{pmatrix} \cdot \begin{pmatrix} -\sin x & \cos y \\ \cos x & -\sin y \end{pmatrix}$$

$$= \begin{pmatrix} -\sin x\,(2\cos(x) + 2\sin y) + \cos(x)\,(2\sin(x) + 2\cos(y)) \\ \cos y\,(2\cos x + 2\sin y) - \sin(y)\,(2\sin(x) + 2\cos y) \end{pmatrix}^t$$

$$= \begin{pmatrix} -\cancel{2\sin x \cos x} - 2\sin x \sin y + \cancel{2\cos x \sin x} + 2\cos x \cos y \\ 2\cos x \cos y + \cancel{2\cos y \sin y} - 2\sin x \sin y - \cancel{2\sin y \cos y} \end{pmatrix}^t$$

$$= \begin{pmatrix} \cancel{-\cos(x-y)} + \cos(x+y) + \cancel{\cos(x-y)} + \cos(x+y) \\ \cancel{\cos(x-y)} + \cos(x+y) - \cancel{\cos(x-y)} + \cos(x+y) \end{pmatrix}^t$$

$$= \begin{pmatrix} 2\cos(x+y) & 2\cos(x+y) \end{pmatrix}$$