Chapter 1

Algebraic sets

1.1 Polynomial equations

Let k be a field.

Definition 1.1. The *affine space of dimension* n is the set k^n.

Definition 1.2. An *algebraic subset* of k^n is a subset $V \subseteq k^n$ for which there exists a subset $A \subseteq k[x_1, \ldots, x_n]$ such that

$$V = \{x \in k^n \mid \forall P \in A \colon P(x) = 0\}.$$

Notation: $V = \mathcal{V}_{k^n}(A)$.

Remark 1.3. If $A \subseteq k[x_1, \ldots, x_n]$ is a subset and I is the ideal generated by A, then

$$\mathcal{V}(A) = \mathcal{V}(I).$$

Definition 1.4. Let $Z \subseteq k^n$ be a subset. Define the ideal in $k[x_1, \ldots, x_n]$

$$\mathcal{I}(Z) \coloneqq \{P \in k[x_1, \ldots, x_n] \mid \forall x \in Z \colon P(x) = 0\}.$$

Remark 1.5. Since $k[x_1, \ldots, x_n]$ is a Noetherian ring, all ideals are finitely generated. For $I \subseteq k[x_1, \ldots, x_n]$ there exist polynomials $P_1, \ldots, P_m \in k[x_1, \ldots, x_n]$ such that $I = (P_1, \ldots, P_m)$ and

$$\mathcal{V}(I) = \mathcal{V}(P_1, \ldots, P_m) = \mathcal{V}(P_1) \cap \ldots \cap \mathcal{V}(P_m).$$

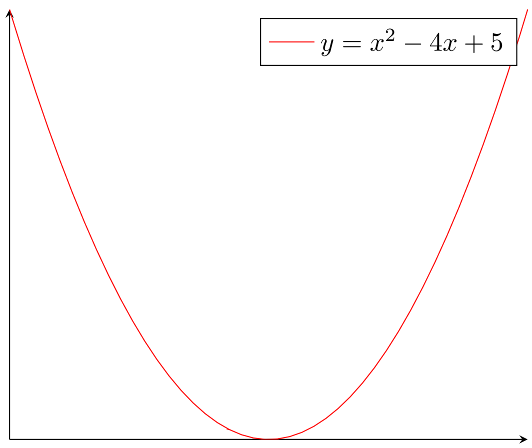

Figure 1.1: parabola

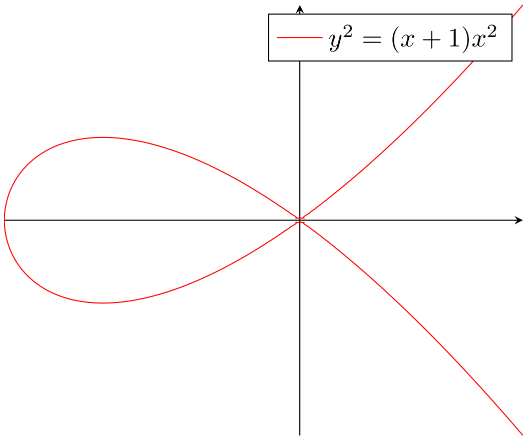

Figure 1.2: nodal cubic

Thus all algebraic subsets of k^n are intersections of hypersurfaces.

Proposition 1.6. *The maps*

$$\mathcal{I}\colon \{\textit{subsets of } k^n\} \longrightarrow \{\textit{ideals in } k[x_1, \ldots, x_n]\}$$

and

$$\mathcal{V}\colon \{\textit{ideals in } k[x_1, \ldots, x_n]\} \longrightarrow \{\textit{subsets of } k^n\}$$

satisfy the following properties

(i) $Z_1 \subseteq Z_2 \implies \mathcal{I}(Z_1) \supseteq \mathcal{I}(Z_2)$

(ii) $I_1 \subseteq I_2 \implies \mathcal{V}(I_1) \supseteq \mathcal{V}(I_2)$

(iii) $\mathcal{I}(Z_1 \cup Z_2) = \mathcal{I}(Z_1) \cap \mathcal{I}(Z_2)$

(iv) $\mathcal{I}(\mathcal{V}(I)) \supseteq I$

(v) $\mathcal{V}(\mathcal{I}(Z)) \supseteq Z$ *with equality if and only if* Z *is an algebraic set.*

Proof. Calculation. □

Lemma 1.7. *Let* $I, J \subseteq k[x_1, \ldots, x_n]$ *be ideals. Then*

$$\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(IJ)$$

where IJ *is the ideal generated by the products* PQ*, where* $P \in I$ *and* $Q \in J$*.*

Lemma 1.8. *Let* $I_j \in k[x_1, \ldots, x_n]$ *be ideals. Then*

$$\bigcap_{j \in J} \mathcal{V}(I_j) = \mathcal{V}\left(\bigcup_{j \in J} I_j\right).$$

1.2 The Zariski topology

The algebraic subsets of k^n can be used to define a topology on k^n.

Proposition 1.9. *The algebraic subsets of k^n are exactly the closed sets of a topology on k^n.*

Proof. $\emptyset = \mathcal{V}(1)$ and $k^n = \mathcal{V}(0)$. The rest follows from 1.7 and 1.8. □

Definition 1.10. The topology on k^n where the closed sets are exactly the algebraic subsets of k^n, is called the *Zariski topology.*

Lemma 1.11. *(i) Let $Z \subseteq k^n$ be a subset. Then*

$$\overline{Z} = \mathcal{V}(\mathcal{I}(Z)).$$

(ii) Let $Z \subseteq k^n$ be a subset. Then

$$\sqrt{\mathcal{I}(Z)} = \mathcal{I}(Z).$$

(iii) Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then

$$\mathcal{V}(I) = \mathcal{V}(\sqrt{I}).$$

Proof. (i) Let $V = \mathcal{V}(I)$ be a Zariski-closed set such that $Z \subseteq V$. Then $\mathcal{I}(Z) \supseteq \mathcal{I}(V)$. But $\mathcal{I}(V) = \mathcal{I}(\mathcal{V}(I)) \supseteq I$, so $\mathcal{V}(\mathcal{I}(Z)) \subseteq \mathcal{V}(I) = V$. Thus

$$\mathcal{V}(\mathcal{I}(Z)) \subseteq \bigcap_{V \text{ closed}, Z \subseteq V} V = \overline{Z}.$$

Since $\mathcal{V}(I(Z))$ is closed, the claim follows.

□

Corollary 1.12. *For ideals $I, J \subseteq k[x_1, \ldots, x_n]$ we have*

$$\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(IJ) = \mathcal{V}(I \cap J).$$

Proof. $\sqrt{I \cap J} = \sqrt{IJ}$ □

Proposition 1.13. *The Zariski topology turns k^n into a Noetherian topological space: If $(F_n)_{n \in N}$ is a decreasing sequence of closed sets, then $(F_n)_{n \in \mathbb{N}}$ is stationary.*

Proof. Let $V_1 \supseteq V_2 \supseteq \ldots$ be a decreasing sequence of closed sets. Then $\mathcal{I}(V_1) \subseteq \mathcal{I}(V_2) \subseteq \ldots$ is an increasing sequence of ideals in $k[x_1, \ldots, x_n]$. As $k[x_1, \ldots, x_n]$ is Noetherian, this sequence is stationary. Thus there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\mathcal{I}(V_n) = \mathcal{I}(V_{n_0})$. Therefore,

$$V_n = \mathcal{V}(\mathcal{I}(V_n)) = \mathcal{V}(\mathcal{I}(V_{n_0})) = V_{n_0}$$

for $n \geq n_0$. □

Definition 1.14. Let $P \in k[x_1, \ldots, x_n]$. The subset

$$D_{k^n}(P) := k^n \setminus \mathcal{V}(P)$$

is called a *standard* or *principal open set* of k^n.

Remark 1.15. Since a Zariski-closed subset of k^n is an intersection of finitely many $\mathcal{V}(P_i)$, a Zariski-open subset of k^n is a union of finitely many standard open sets. Thus the standard open sets form a basis for the Zariski topology of k^n.

Proposition 1.16. *The affine space k^n is quasi-compact in the Zariski topology.*

Proof. Let $k^n = \bigcup_{i \in J} U_i$ where U_i is open. Since the standard opens form a basis of the Zariski topology, we can assume $U_i = D(P_i)$ with $P_i \in k[x_1, \dots, x_n]$. Then $\mathcal{V}((P_i)_{i \in I}) = \bigcap_{i \in J} \mathcal{V}(P_i) = \emptyset$. Since $k[x_1, \dots, x_n]$ is Noetherian, we can choose finitely many generators $P_{i_1}, \dots, P_{i_m}$ such that $((P_i)_{i \in J}) = (P_{i_1}, \dots, P_{i_m})$. Thus

$$\bigcap_{j=1}^{m} \mathcal{V}(P_{i_j}) = \mathcal{V}(P_{i_1}, \dots, P_{i_m}) = \mathcal{V}((P_i)_{i \in J}) = \emptyset.$$

By passing to complements in k^n, we get

$$\bigcup_{j=1}^{m} D(P_{i_j}) = k^n.$$

□

Proposition 1.17. *Let $P \in k[x_1, \dots, x_n]$ and let $f_P \colon k^n \to k$ be the associated function on k^n. Then f_p is continuous with respect to the Zariski topology on k^n and k.*

Proof. The closed proper subsets of k are finite subsets $F = \{t_1, \dots, t_s\} \subseteq k$. The pre-image of a singleton $\{t\} \subseteq k$ is

$$f_P^{-1}(\{t\}) = \{x \in k^n \mid P(x) - t = 0\} = \mathcal{V}(P - t)$$

which is a closed subset of k^n. Thus

$$f_P^{-1}(F) = \bigcup_{i=1}^{s} \mathcal{V}(P - t_i)$$

is closed. □

Proposition 1.18. *If k is infinite, $\mathcal{I}(k^n) = \{0\}$.*

Proof. By induction: for $n = 1$, this follows because a non-zero polynomial only has a finite number of roots. Let $n \geq 1$ and $P \in \mathcal{I}(k^n)$. Thus $P(x) = 0 \ \forall x \in k^n$. Let

$$P = \sum_{i=0}^{m} P_i(X_1, \dots, X_{n-1}) X_n^i$$

for $P_i \in k[X_1, \dots, X_{n-1}]$. Fix some $x_1, \dots, x_{n-1} \in k$. Then $P(x_1, \dots, x_{n-1}, y) \in k[y]$ has an infinite number of roots. Thus $P(x_1, \dots, x_{n-1}, y) = 0$ for all $x_2, \dots, x_n$ by the case $n = 1$, implying that $P_i(x_1, \dots, x_{n-1}) = 0$ for all i. Since this holds for all $(x_1, \dots, x_{n-1}) \in k^{n-1}$, $P_i = 0$ by induction for all i. □