Chapter 1

Real algebra

1.1 Ordered fields and real fields

Definition 1.1. An *ordered field* is a pair $(k, \leq)$ consisting of a field k and an order relation $\leq$ such that

(i) $\leq$ is a total order: if $x, y \in k$, then $x \leq y$ or $y \leq x$.

(ii) $\leq$ is compatible with addition in k: if $x, y, z \in k$, then $x \leq y$ implies $x + z \leq y + z$.

(iii) $\leq$ is compatible with multiplication in k: if $x, y \in k$, then $0 \leq x$ and $0 \leq y$ implies $0 \leq xy$.

A morphism between two ordered fields $(k, \leq)$ and $(L, \leq)$ is a field homomorphism $\varphi\colon k \to L$ such that $x \leq y$ in k implies $\varphi(x) \leq \varphi(y)$ in L.

Example 1.2. (1) The fields $\mathbb{Q}$ and $\mathbb{R}$, equipped with their usual orderings, are ordered fields.

(2) The field $\mathbb{C}$ can be equipped with a total ordering (the „lexicographic order“) but not with a structure of ordered field.

(3) The field $\mathbb{R}(t)$ of rational fractions with coefficients in $\mathbb{R}$, can be equipped with a structure of ordered field in multiple ways:

Fix an $x \in \mathbb{R}$ and, for all polynomial $P \in \mathbb{R}[t]$, use Taylor expansion at x to write

$$P(t) = a_p(t - x)^p + \text{higher order terms.}$$

with $a_p \neq 0$, then define $P(t) >_{x^+} 0$ if $a_p > 0$, i.e. if the function $t \mapsto P(t)$ is positive on a small interval $(x, x + \varepsilon)$. Set also $\frac{P(t)}{Q(t)} >_{x^+} 0$ if $P(t)Q(t) >_{x^+} 0$, and define $f \leq_{x^+} g$ in $\mathbb{R}(t)$ if either $f = g$ or $g - f >_{x^+} 0$. Equivalently $f \leq_{x^+} g$ in $\mathbb{R}(t)$ if either $f = g$ or $g - f$ is positively-valued on $(x, x + \varepsilon)$ for $\varepsilon > 0$ small enough.

It is clear that this is a total ordering on $\mathbb{R}(t)$, and that this ordering is compatible with addition and multiplication in the sense of the definition of an ordered field. Moreover, the substitution homomorphism $h(t) \mapsto h(t - x)$ induces an isomorphism of ordered fields $(\mathbb{R}(t), \leq_{0^+}) \xrightarrow{\simeq} (\mathbb{R}(t), \leq_{x^+})$, since a function $t \mapsto h(t - x)$ is positively-valued on $(x, x + \varepsilon)$ if and only if the function $t \mapsto h(t)$ is positively valued on $(0, \varepsilon)$.

Note that we can also define orderings on $\mathbb{R}(t)$ by setting $f \leq_{x^-} g$ if either $f = g$ or $g - f$ is positively-valued on $(x - \varepsilon, x)$, for $\varepsilon > 0$ small enough. The substitution homomorphism $h(t) \mapsto h(-t)$ induces an isomorphism of ordered fields $(\mathbb{R}(t), \leq_{0^-}) \xrightarrow{\simeq} (\mathbb{R}(t), \leq_{0^+})$.

Remark 1.3. The ordered field $(\mathbb{R}(t), \leq_{0^+})$ is non-Archimedean: the element t is *infinitely small with respect to any real* $\delta > 0$ in the sense that for all $n \in \mathbb{N}$, $nt < \delta$ (indeed $t \mapsto nt - \delta$ is negatively-valued on $(0, \varepsilon)$ for $\varepsilon > 0$ small enough). Equivalently, $\frac{1}{t}$ is infinitely large with respect to $0 < \delta \in \mathbb{R}$ in the sense that $\frac{1}{t} > n\delta$ for all $n \in \mathbb{N}$.

Proposition 1.4. *Let $(k, \leq)$ be an ordered field and $x, y, z \in k$. Then the following properties hold:*

(a) $x \geq 0$ *or* $-x \geq 0$.

(b) $-1 < 0$ *and* $1 > 0$.

(c) k *is of characteristic* 0.

(d) if $x < y$ *and* $z > 0$, *then* $xz < yz$.

(e) if $x < y$ *and* $z < 0$, *then* $xz > yz$.

(f) $xy \geq 0$ *if and only if* x *and* y *have the same sign.*

(g) $x^2 \geq 0$ *and, if* $x \neq 0$, *then* x *and* $\frac{1}{x}$ *have the same sign.*

(h) if $0 < x \leq y$, *then* $0 < \frac{1}{y} \leq \frac{1}{x}$.

Proof. Elementary verifications. □

It turns out that it is possible to characterise ordered fields without explicitly mentioning the order relation, using cones of positive elements.

Definition 1.5. Let k be a field. A *cone* in k is a subset $P \subseteq k$ such that for all $x, y \in P$ and $z \in k$:

(i) $x + y \in P$

(ii) $xy \in P$

(iii) $z^2 \in P$

A cone $P \subseteq k$ is called a *positive cone* if, additionally, one has:

(iv) $-1 \notin P$

Proposition 1.6. *Let k be a field. Assume that there exists a positive cone $P \subseteq k$. Then:*

(i) $0 \in P$ *and* $1 \in P$.

(ii) k *is of characteristic* 0.

(iii) $P \cap (-P) = \{0\}$

Proof. (i) $0 = 0^2 \in P$ and $1 = 1^2 \in P$ by axiom (iii).

(ii) Since $1 \in P$, by induction and axiom (i), $n \cdot 1 = \underbrace{1 + \ldots + 1}_{n \text{ times}} \in P$ for all $n \in \mathbb{N}$. Assume that there exists $n \in \mathbb{N}$, such that $n \cdot 1 = 0$ in k. Since $1 \neq 0$ in k, it follows $n \geq 2$ so,

$$-1 = 0 - 1 = n \cdot 1 - 1 = (n - 1) \cdot 1 \in P,$$

which contradicts axiom (iv).

(iii) Assume that there exists $x \in P \cap (-P) \setminus \{0\}$. In particular $x \neq 0$ and $-x \in P$. So $-x^2 = (-x)x \in P$ by axiom (ii) and $\frac{1}{x^2} = \left(\frac{1}{x}\right)^2 \in P$ by axiom (iii). Again by axiom (ii)

$$-1 = \frac{1}{x^2}(-x^2) \in P$$

which contradicts axiom (iv).

□

Given a positive cone P in a field k, let us set $P^+ = P \setminus \{0\}$ and $P^- = (-P) \setminus \{0\} = -P^+$. Then we have a disjoint union

$$P^- \sqcup \{0\} \sqcup P^+ \subseteq k.$$

Note that P^+ satisfies axioms (i) and (ii) of the definition of a cone, as well as the property that $x \in k \setminus \{0\} \implies x^2 \in P^+$.

We now prove that positive curves can be enlarged, that the resulting notion of maximal positive cone satisfies $P \cup (-P) = k$, and that this defines a structure of ordered field on k by setting $x \leq y$ if and only if $y - x \in P$.

Lemma 1.7. *Assume that P is a positive cone in a field k. If $a \in k \setminus P \cup (-P)$, then the set*

$$P[a] \coloneqq \{x + ay \in k \colon x, y \in P\}$$

is a positive cone in k, satisfying $P \subsetneq P[a]$.

Proof. Let $x, y, x', y' \in P$. Then

$$(x + ay) + (x' + ay') = x + x' + a(y + y') \in P[a]$$

and

$$(x + ay)(x' + ay') = xx' + a^2yy' + a(xy' + x'y) \in P[a].$$

Moreover $z^2 \in P \subseteq P[a]$ for all $z \in k$.

Now assume $-1 = x + ay$ for some $x, y \in P$. If $y = 0$, then $-1 = x \in P$ which is a contradiction. Thus $y \neq 0$ and

$$-a = \frac{1 + x}{y} = \left(\frac{1}{y}\right)^2 y(1 + x) \in P,$$

which contradicts the assumption on a. Finally, we have $P \subseteq P[a]$ and, if $P[a] \subseteq P$ then $a \in P$, again contradicting the assumption on a. So $P \subsetneq P[a]$. □

Proposition 1.8. *Let $\mathcal{P}$ be the set of positive cones of a field k ordered by inclusion. If $\mathcal{P} \neq \emptyset$, then $\mathcal{P}$ admits a maximal element and such an element P satisfies $P \cup (-P) = k$.*

Proof. To obtain a maximal element of $\mathcal{P}$, by Zorn's lemma, it suffices to show, that every chain $(P_i)_{i \in I}$ in $\mathcal{P}$ has an upper bound. We set

$$P = \bigcup_{i \in I} P_i \subseteq k.$$

One verifies immediately that P is a positive cone and an upper bound of the chain $(P_i)_{i \in I}$.

Let P be such a maximal element. If there exists $a \in k \setminus P \cup (-P)$, then by 1.7 $P \subsetneq P[a]$ contradicts the maximality of P. Thus $P \cup (-P) = k$. □

Proposition 1.9. *Let k be a field and denote by*

$$\Sigma k^{[2]} \coloneqq \left\{ y \in k \mid \exists (a_x)_{x \in k} \in \{0, 1\}^{(k)}, y = \sum_{x \in k} a_x x^2 \right\}$$

the set of sums of squares in k. Then $\Sigma k^{[2]}$ is a cone and $-1 \notin \Sigma k^{[2]}$ if and only if for all $x_1, \ldots, x_n \in k$:

$$x_1^2 + \ldots + x_n^2 = 0 \implies x_1 = \ldots = x_n = 0.$$

Proof. One verifies immediately that $\Sigma k^{[2]}$ is a cone in k. If $-1 \in \Sigma k^{[2]}$, then $-1 = x_1^2 + \ldots + x_n^2$ for some $x_i \in k$. Thus

$$0 = \sum_{i=1}^{n} x_i^2 + 1$$

but $1 = 1^2$ and $1 \neq 0$. Conversely let $0 = \sum_{i=1}^{n} x_i^2$ with $x_1 \neq 0$. Then

$$-1 = \frac{1}{x_1^2} \sum_{i=2}^{n} x_i^2 = \sum_{i=2}^{n} \left(\frac{x_i}{x_1} \right)^2 \in \Sigma k^{[2]}.$$

□

Definition 1.10. A field k is called a *real field* if $-1 \notin \Sigma k^{[2]}$, or equivalently if $\sum_{k=1}^{n} x_i^2 = 0$ in k implies $x_k = 0$ for all k.

Corollary 1.11. *Let k be a field. k is real if and only if k contains a positive cone.*

Proof. ($\Rightarrow$): By 1.9 $\Sigma k^{[2]}$ is a positive cone. ($\Leftarrow$): Let P be a positive cone. Since P is closed under addition and for all $z \in k$: $z^2 \in P$, $\Sigma k^{[2]} \subset P$. Since P is positive, $-1 \notin \Sigma k^{[2]}$. □

Proposition 1.12. *Let $(k, \leq)$ be an ordered field. Then the set*

$$P \coloneqq \{x \in k \mid x \geq 0\}$$

is a maximal positive cone in k. In particular, k is a real field. Conversely, if k is a real field and P is a maximal positive cone in k, then the relation $x \leq_P y$ if $y - x \in P$ is an order relation and $(k, \leq_P)$ is an ordered field.

Proof. ($\Rightarrow$): Let $(k, \leq)$ be an ordered field. Then by definition and 1.4, P is a maximal positive cone.

($\Leftarrow$): Let P be a maximal positive cone in k. Since $0 \in P$, we have $x \leq_P x$. Suppose that $x \leq_P y$ and $y \leq_P x$. Then $y - x \in P \cap (-P) = \{0\}$, so $x = y$. Moreover, if $x \leq_P y$ and $y \leq_P z$, then $z - x = (z - y) + (y - x) \in P$. Thus $x \leq_P z$, hence $\leq_P$ is an order relation. Moreover, it is a total order, because if $x, y \in k$, then $y - x \in k = P \cup (-P)$, so either $x \leq_P y$ or $y \leq_P x$.

Finally, this total order on k is compatible with addition and multiplication because $x \leq_P y$ and $z \in k$ implies $(y + z) - (x + z) = y - x \in P$, so $x + z \leq_P y + z$, and $x \geq_P 0$, $y \geq_P 0$ means that $x \in P$ and $y \in P$, so $xy \in P$, hence $xy \geq_P 0$. □

Corollary 1.13. *Let k be a field. Then k admits a structure of ordered field if and only if k is real.*