0.1 Real-closed fields

In this section we study real algebraic extensions of real fields.

Lemma 0.1. *Let k be a real field and $x \in k \setminus \{0\}$. Then x and $-x$ cannot be both sums of squares in k.*

Proof. If $x \in \Sigma k^{[2]}$ and $-x \in \Sigma k^{[2]}$, then

$$1 = \frac{1}{x^2}(-x)x \in \Sigma k^{[2]}$$

contradicting that k is real. □

Proposition 0.2. *Let k be a real field and $a \in k$ such that a is not a square in k. Then the field*

$$k(\sqrt{a}) = k[t]/(t^2 - a)$$

is real if and only if $-a \notin \Sigma k^{[2]}$. In particular, if $\Sigma k^{[2]} \cup (-\Sigma k^{[2]}) \neq k$, then k admits real quadratic extensions.

Proof. Since a is not a square in k, $t^2 - a$ is irreducible in $k[t]$, so $k[t]/(t^2 - a)$ is indeed a field. Denote by $\sqrt{a}$ the class of t in the quotient.

($\Rightarrow$): a is a square in $k(\sqrt{a})$, thus by 0.1 we have $-a \notin \Sigma k(\sqrt{a})^2$. But $\Sigma k^{[2]} \subseteq \Sigma k(\sqrt{a})^{[2]}$, thus $-a \notin \Sigma k(\sqrt{a})^{[2]}$.

($\Leftarrow$): $-1 \in \Sigma k(\sqrt{a})^{[2]}$ if and only if there exist $x_i, y_i \in k$, such that

$$-1 = \sum_{i=1}^{n}(x_i + y_i\sqrt{a})^2 = \sum_{i=1}^{n}(x_i^2 + ay_i^2) + 2\sqrt{a}\sum_{i=1}^{n} x_i y_i.$$

Since $(1, \sqrt{a})$ is a basis of the k-vector space $k(\sqrt{a})$, the previous equality implies

$$-1 = \sum_{i=1}^{n} x_i^2 + a\sum_{i=1}^{n} y_i^2.$$

Since $-1 \notin \Sigma k^{[2]}$, $\sum_{i=1}^{n} y_i^2 \neq 0$, this implies

$$-a = \frac{1 + \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} y_i^2} = \frac{\left(\sum_{i=1}^{n} y_i^2\right)\left(1 + \sum_{i=1}^{n} x_i^2\right)}{\left(\sum_{i=1}^{n} y_i^2\right)^2} \in \Sigma k^{[2]}.$$

□

Simple extensions of odd degree are simpler from the real point of view:

Proposition 0.3. *Let k be a real field and $P \in k[t]$ be an irreducible polynomial of odd degree. Then the field $k[t]/(P)$ is real.*

Proof. Denote by n the degree of P. We proceed by induction on $n \geq 1$. If $n = 1$, then $k[t]/(P) \simeq k$ is real. Since n is odd, we may now assume $n \geq 3$. Let $L := k[t]/(P)$. Suppose L is not real. Then there exist polynomials $g_i \in k[t]$, of degree at most $n-1$, such that $-1 = \sum_{i=1}^{m} g_i^2$ in $L = k[t]/(P)$. Since $k \subseteq L$ and k is real, at least one of the g_i is non-constant. By definition of L, there exists $Q \in k[t] \setminus \{0\}$ such that

$$-1 = \sum_{i=1}^{m} g_i^2 + PQ \tag{1}$$

in $k[t]$. Since k is real, in $\sum_{i=1}^{m} g_i^2$ no cancellations of the terms of highest degree can occur. Thus $\sum_{i=1}^{m} g_i^2$ is of positive, even degree at most $2n-2$. By 1, it follows that Q is of odd degree at most $n-2$. In particular, Q has at least one irreducible factor Q_1 of odd degree at most $n-2$. Since $n \geq 3$, $n-2 \geq 1$. By induction, $M := k[t]/(Q_1)$ is real. But 1 implies

$$-1 = \sum_{i=1}^{m} g_i^2$$

in $M = k[t]/(Q_1)$ contradicting the fact that M is real. □

Definition 0.4. A *real-closed* field is a real field that has no proper real algebraic extensions.

Theorem 0.5. *Let k be a field. Then the following conditions are equivalent:*

(i) k is real-closed.

(ii) k is real and for all $a \in k$, either a or $-a$ is a square in k and every polynomial of odd degree in $k[t]$ has a root in k.

(iii) the k-algebra

$$k[i] := k[t]/(t^2+1)$$

is algebraically closed.

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Proof. (i)⇒(ii): Let $a \in k$ such that neither a nor $-a$ is a square in k. Then by 0.2 and (i), $\pm a \in \Sigma k^{[2]}$ contradicting 0.1. Let $P \in k[t]$ be a polynomial of odd degree. P has at least one irreducible factor P_1 of odd degree. By 0.3, $k[t]/(P_1)$ is a real extension of k. Since k is real-closed, P_1 must be of degree 1 and thus P has a root in k.

(ii)⇒(iii): Since -1 is not a square in k, the polynomial t^2+1 is irreducible over k. Thus $L := k[t]/(t^2+1)$ is a field. Denote by i the image of t in L and for $x = a+ib \in L = k[i]$, denote by $\overline{x} = a - ib$. This extends to a ring homomorphism $L[t] \to L[t]$. Let $P \in L[t]$ be non-constant. It remains to show, that P has a root in L. We first reduce to the case $P \in k[t]$.

Assume every non-constant polynomial in $k[t]$ has a root in L. Let $P \in L[t]$. Then $P\overline{P} \in k[t]$ has a root $x \in L$, thus either $P(x) = 0$ or $\overline{P}(x) = 0$. In the first case, we are done. In the second case, we have $P(\overline{x}) = \overline{\overline{P}(x)} = \overline{0} = 0$, so $\overline{x}$ is a root of P in L.

Thus we may assume $P \in k[t]$. Write $d = \deg(P) = 2^m n$ with $2 \nmid n$. We proceed by induction on m. If $m = 0$, the result is true by (ii). Now assume $m > 0$. Fix an algebraic closure $\overline{k}$ of k. Since k is real, it is of characteristic 0, thus k is perfect and $\overline{k}/k$ is galois. Let $y_1, \ldots, y_d$ be the roots of P in $\overline{k}$. Consider for all $r \in \mathbb{Z}$:

$$F_r := \prod_{1 \leq p < q \leq d} (t - (y_p + y_q) - r y_p y_q)) \in \overline{k}[t].$$

This polynomial with coefficients in $\overline{k}$ is invariant under permutation of $y_1, \ldots, y_d$. Thus its coefficients lie in $\overline{k}^{\mathrm{Gal}(\overline{k}/k)} = k$. Moreover

$$\deg(F_r) = \binom{d}{2} = \frac{d(d-1)}{2} = 2^{m-1} n (2^m - 1).$$

with $n(2^m-1)$ odd. So the induction hypothesis applies and, for all $r \in \mathbb{Z}$, there is a pair $p < q$ in $\{1, \ldots, d\}$ such that $(y_p + y_q) + r y_p y_q \in L$. Since $\mathbb{Z}$ is infinite, we can find a pair $p < q$ in $\{1, \ldots, d\}$ for which there exists a pair $r \neq r'$ such that

$$(y_p + y_q) + r y_p y_q \in L$$
$$\text{and } (y_p + y_q) + r' y_p y_q \in L.$$

By solving the system, we get $y_p + y_q \in L$ and $y_p y_q \in L$. But y_p, y_q are roots of the quadratic polynomial

$$t^2 - (y_p + y_q)t + y_p y_q \in L[t]$$

and since $i^2 = -1$, the roots of this polynomial lie in $L = k[i]$, by (ii) and the usual formulas

$$t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So P indeed has a root in $k[i]$, which finishes the induction.

(iii)⇒(i): Denote again by i the image in the algebraically closed field $k[t]/(t^2+1)$. We first show that $k^{[2]} = \Sigma k^{[2]}$. Let $a, b \in k$. Then $a + ib = (c + id)^2$ in $k[i]$ for some $c, d \in k$. Thus

$$a^2 + b^2 = (a + ib)(a - ib) = (c + id)^2 (c - id)^2 = (c^2 + d^2)^2.$$

By induction the claim follows. Since $t^2 + 1$ is irreducible, $-1 \notin k^{[2]} = \Sigma k^{[2]}$ and k is real.

Let L be a real algebraic extension of k. Since $k[i]$ is algebraically closed and contains k, there exists a k-homomorphism $L \hookrightarrow k[i]$. Since $[k[i] : k] = 2$, either $L = k$ or $L = k[i]$, but $k[i]$ is not real, since $i^2 = -1$ in $k[i]$. So $L = k$ and k is real-closed. □

Corollary 0.6. *A real-closed field k admits a canonical structure of ordered field, in which the cone of positive elements is exactly $k^{[2]}$, the set of squares in k.*

Proof. This was proven in the implication (i)⇒(ii) of 0.5. □

Example 0.7.
- $\mathbb{R}$ is a real-closed field, because $\mathbb{R}[i] = \mathbb{C}$ is algebraically closed.
- The field of real Puiseux series

$$\widehat{\mathbb{R}(t)} := \bigcup_{q>0} \mathbb{R}((t^{\frac{1}{q}})) = \left\{ \sum_{n=m}^{\infty} a_n t^{\frac{n}{q}} : m \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\}, a_n \in \mathbb{R} \right\}$$

is a real closed field because $\widehat{\mathbb{R}(t)}[i] = \widehat{\mathbb{R}[i][t]} = \widehat{\mathbb{C}[t]}$ is the field of complex Puiseux series, which is algebraically closed by the Newton-Puiseux theorem.

Remark 0.8. By 0.5, if k is a real-closed field, then the absolute galois group of k is

$$\mathrm{Gal}(\overline{k}/k) = \mathrm{Gal}(k[i]/k) \simeq \mathbb{Z}/2\mathbb{Z}.$$

The Artin-Schreier theorem shows that if $\overline{k}/k$ is a non-trivial extension of *finite* degree, then k is real-closed.