0.1 Extensions of ordered fields

If a field k admits a structure of ordered field, we will say that k is *orderable*. For an *ordered* field k, an extension L/k is called *oderable* if the field L is orderable such that the induced order on k coincides with the fixed order on k.

Definition 0.1. Let k be a field. A quadratic form $q\colon k^n \to k$ is called *isotropic* if there exists $x \in k \setminus \{0\}$ such that $q(x) = 0$. Otherwise, the quadratic form is called *anisotropic.*

Remark 0.2. Recall that, given a quadratic form q on a finite-dimensional k-vector space E, there always exists a basis of E in which $q(x_1, \ldots, x_n) = a_1x_1^2 + \ldots + a_rx_r^2$, where $r = \mathrm{rg}(q) \leq n = \dim\, E$ and $a_1, \ldots, a_r \in k$. The form q is non-degenerate on E if and only if $r = \dim\, E$.

Example 0.3.

- A field k is real if and only if for all $n \in \mathbb{N}$, the form $x_1^2 + \ldots + x_n^2$ is anisotropic.
- A degenerate quadratic form is isotropic.
- If k is algebraically closed and $n \geq 2$, all quadratic forms on k^n are isotropic.
- If $(k, \leq)$ is an ordered field and $q(x_1, \ldots, x_n) = a_1x_1^2 + \ldots + a_nx_n^2$ with $a_i > 0$ for all i, then q and $-q$ are anisotropic on k^n.

Definition 0.4. Let k be a field and L an extension of k. A quadratic form $q\colon k^n \to k$ induces a quadratic form $q_L\colon L^n \to L$. The form q is called *anisotropic over* L if q_L is anisotropic.

It can be checked that, on an ordered field $(k, \leq)$, a quadratic form q is anisotropic if and only if it is non-degenerate and of constant sign. The interest of this notion for us is given by the following result.

Theorem 0.5. *Let $(k, \leq)$ be an ordered field and L be an extension of k. Then the following conditions are equivalent:*

(i) The extension L/k is orderable.

(ii) For all $n \geq 1$ and all $a = (a_1, \ldots, a_n) \in k^n$ such that $a_i > 0$ for all i, the quadratic form $q(x_1, \ldots, x_n) = a_1x_1^2 + \ldots + a_nx_n^2$ is anisotropic over L (i.e. all positive definite quadratic forms on k are anisotropic over L).

Proof. (i)⇒(ii): Assume that there is an ordering of L that extends the ordering of k and let $n \geq 1$. Let $a = (a_1, \ldots, a_n) \in k^n$ with $a_i > 0$ for all i. Then $a_i > 0$ still holds in L. Since squares are non-negative for all orderings, the sum $a_1x_1^2 + \ldots + a_nx_n^2$ is a sum of positive terms in L. Therefore it can only be 0, if all of its terms are 0. Since $a_i \neq 0$, it follows $x_i = 0$ for all i.

(ii)⇒(i): Define

$$P = \bigcup_{n \geq 1} \left\{ \sum_{i=1}^{n} a_i x_i^2 \colon a_i \in k, a_i > 0, x_i \in L \right\}.$$

The set P is stable by sum and product and contains all squares of L, so it is a cone in L. Suppose $-1 \in P$. Then there exists $n \geq 1$ and $a = (a_1, \ldots, a_n) \in k^n$ with $a_i > 0$ and $x = (x_1, \ldots, x_n) \in L^n$ such that $-1 = \sum_{i=1}^n a_i x_i^2$. So

$$a_1x_1^2 + \ldots + a_nx_n^2 + 1 = 0,$$

meaning that the quadratic form $a_1x_1^2 + \ldots + a_nx_n^2 + x_{n+1}^2$ is isotropic on L^{n+1}, contradicting (ii). Thus P is a positive cone containing all positive elements of k. By embedding P in a maximal positive cone, the claim follows. □

Proposition 0.6. *Let $(k, \leq)$ be an ordered field and let $c > 0$ be a positive element in k. Then $k[\sqrt{c}]$ is an orderable extension of k.*

Proof. If c is a square in k, there is nothing to prove. Otherwise, $k[\sqrt{c}]$ is indeed a field. Let $n \geq 1$ and let $a = (a_1, \ldots, a_n) \in k^n$ with $a_i > 0$ for all i. Assume that $x = (x_1, \ldots, x_n) \in k[\sqrt{c}]^n$ satisfies

$$a_1 x_1^2 + \ldots + a_n x_n^2 = 0.$$

Since $x_i = u_i + v_i\sqrt{c}$ for some $u_i, v_i \in k$, we can rewrite this equation as

$$\sum_{i=1}^{n} a_i(u_i^2 + cv_i^2) + 2\sum_{i=1}^{n} u_i v_i \sqrt{c} = 0.$$

Since 1 and $\sqrt{c}$ are linearly independent over k, we get $\sum_{i=1}^{n} a_i(u_i^2 + cv_i^2) = 0$, hence $u_i = v_i = 0$ for all i, since all terms in the previous sum are non-negative. So $x_i = 0$ for all i and (ii) of 0.5 is satisfied. □

Proposition 0.7. *Let $(k, \leq)$ be an ordered field and let $P \in k[t]$ be an irreducible polynomial of odd degree. Then the field $L := k[t]/(P)$ is an orderable extension of k.*

Proof. Denote by d the degree of P and proceed by induction on $d \geq 1$. If $d = 1$, then $L = k$. Now assume $d \geq 2$. Let $n \geq 1$ and $a_1, \ldots, a_n \in k$ with $a_i > 0$. Denote by q_L the quadratic form

$$q_L(x_1, \ldots, x_n) = a_1 x_1^2 + \ldots + a_n x_n^2$$

on L^n. If q_L is isotropic over L, then there exist polynomials $g_1, \ldots, g_n \in k[t]$ with $\deg(g_i) < d$ and $h \in k[t]$ such that

$$q_L(g_1, \ldots, g_n) = hP \tag{1}$$

Let g be the greatest common divisor of $g_1, \ldots, g_n$. Since q_L is homogeneous of degree 2, g^2 divides $q_L(g_1, \ldots, g_n)$. Since P is irreducible, g divides h. We may thus assume that $g = 1$. The leading coefficients of the terms on the left hand side of (1) are non-negative, thus the sum has even degree $< 2d$. Since the degree of P is odd, h must be of odd degree $< d$. Therefore, h has an irreducible factor $h_1 \in k[t]$ of odd degree. Let α be a root of h_1. By evaluating (1) at α, we get

$$q_{k[\alpha]}(g_1(\alpha), \ldots, g_n(\alpha)) = 0$$

in $k[\alpha]$. Since the $gcd(g_1, \ldots, g_n) = 1$ and $k[t]$ is a principal ideal domain, there exist $h_1, \ldots, h_n \in k[t]$ such that

$$h_1 g_1 + \ldots + h_n g_n = 1.$$

In particular

$$h_1(\alpha)g_1(\alpha) + \ldots + h_n(\alpha)g_n(\alpha) = 1,$$

so not all $g_i(\alpha)$ are 0 in $k[\alpha]$. Thus $q_{k[\alpha]}$ is isotropic over $k[\alpha] = k[t]/(h_1)$ contradicting the induction hypothesis. □