0.1 Real closures

Proposition 0.1. *Let k be a real field. Then there exists a real-closed algebraic orderable extension k^r of k.*

Proof. Let $\overline{k}$ be an algebraic closure of k and E be the set of intermediate extensions $k \subseteq L \subseteq \overline{k}$ such that L is real and algebraic over k. $E \neq \emptyset$ since $k \in E$. Define $L_1 < L_2$ on E if and only if $L_1 \subseteq L_2$ and L_2/L_1 is ordered, i.e. the order relation on L_1 coincides with the on induced by L_2. Then every totally ordered familiy $(E_i)_{i \in I}$ has an upper bound, namely $\bigcup_{i \in I} E_i$. By Zorn, E has a maximal element, which we denote by k^r and which is an algebraic extension of k. Such a k^r is real-closed, because otherwise it would admit a proper real algebraic extension contradicting the maximality of k^r as a real algebraic extension of k. □

Definition 0.2. A real-closed real algebraic extension of a real field k is called a *real closure* of k.

Remark 0.3. By the construction in the proof of 0.1, a real closure of a real field k can be chosen as a subfield k^r of an algebraic closure of $\overline{k}$. Since $k^r[i]$ is algebraically closed and algebraic over k^r, so also over k, it follows $k^r[i] = \overline{k}$.

Proposition 0.4. *Let k be a real field and L be a real-closed extension of k. Let $\overline{k}^L$ be the relative algebraic closure of k in L, i.e.*

$$\overline{k}^L = \{x \in L \mid x \textit{ algebraic over } k\}.$$

Then $\overline{k}^L$ is a real closure of k.

Proof. It is immediate that $\overline{k}^L$ is a real algebraic extension of k. Let $x \in \overline{k}^L$. Then x or $-x$ is a square in L, since L is real-closed. Without loss of generality, assume that $x \in L^{[2]}$. Then $t^2 - x \in \overline{k}^L[t]$ has a root in L. Since this root is algebraic over $\overline{k}^L$, hence over k, it belongs to $\overline{k}^L$. Thus x is in fact a square in $\overline{k}^L$. By the same argument every polynomial of odd degree has a root in $\overline{k}^L$. □

Example 0.5. (i) $\overline{\mathbb{Q}}^{\mathbb{R}} = \overline{\mathbb{Q}}^{\mathbb{C}} \cap \mathbb{R}$ is a real closure of $\mathbb{Q}$. In particular, $\overline{\mathbb{Q}}^{\mathbb{C}} = \overline{\mathbb{Q}}^{\mathbb{R}}[i]$ as subfields of $\mathbb{C}$.

(ii) Consider the real field $k = \mathbb{R}(t)$ and the real-closed extension

$$\widehat{\mathbb{R}(t)} = \bigcup_{q>0} \mathbb{R}((t^{t/q})).$$

Then the subfield $\overline{\mathbb{R}(t)}^{\widehat{\mathbb{R}(t)}}$, consisting of all those real Puiseux series that are algebraic over $\mathbb{R}(t)$, is a real closure of $\mathbb{R}(t)$.

The field of real Puiseux series itself is a real closure of the field $\mathbb{R}((t))$ of real formal Laurent series.

Real-closed fields L admit a canonical structure of ordered field, where $x \geq 0$ in L, if and only if x is a square. In particular, if k is a real field and k^r is a real closure of k, then k inherits an ordering from k^r. However, different real closures may induce different orderings on k, as the next example shows.

Example 0.6. Let $k = \mathbb{Q}(t)$. This is a real field, since $\mathbb{Q}$ is real. Since π is transcendental over $\mathbb{Q}$, we can embed $\mathbb{Q}(t)$ in $\mathbb{R}$ by sending t to π.

$$i_1 : \mathbb{Q}(t) \overset{\simeq}{\hookrightarrow} \mathbb{Q}(\pi) \subseteq \mathbb{R}.$$

Since $\mathbb{R}$ is real-closed, the relative algebraic closure $i_1(\mathbb{Q}(t))^{\mathbb{R}}$ is a real closure of $i_1(\mathbb{Q}(t))$.

We can also embed $\mathbb{Q}(t)$ in the field $\widehat{\mathbb{R}(t)}$ of real Puiseux series via a homomorphism i_2 and then $\overline{i_2(\mathbb{Q}(t))}^{\widehat{\mathbb{R}(t)}}$ is a real closure of $i_2(\mathbb{Q}(t))$. However, the ordering on $\overline{i_1(\mathbb{Q}(t))}^{\mathbb{R}}$ is Archimedean, because it is a subfield of $\mathbb{R}$, while the ordering on $\overline{i_2(\mathbb{Q}(t))}^{\widehat{\mathbb{R}(t)}}$ is not Archimedean (it contains infinitesimal elements, such as t for instance).

The fields $\overline{i_1(\mathbb{Q}(t))}^{\mathbb{R}}$ and $\overline{i_2(\mathbb{Q}(t))}^{\widehat{\mathbb{R}(t)}}$ cannot be isomorphic as fields. Indeed, when two real-closed fields L_1, L_2 are isomorphic as fields, then they are isomorphic as ordered fields, since positivity on a real closed field is defined by the condition of being a square, which is preserved under isomorphisms of fields.

The next result will be proved later on.

Lemma 0.7. *Let $(k, \leq)$ be an ordered field and $P \in k[t]$ be an irreducible polynomial. Let L_1, L_2 be real-closed extensions of k that are compatible with the ordering of k. Then P has the same number of roots in L_1 as in L_2.*

Remark 0.8. In particular, if $P \in k[t]$ is an arbitrary polynomial, then if P has a root in a real-closed extension L of k, then it has a root in all real-closed extensions of k.

A polynomial with coefficients in an ordered field $(k, \leq)$ might not have roots in any real-closed extensions of k.

Lemma 0.9. *Let $(k, \leq)$ be an ordered field, L/k an orderable real-closed extension of k and $\varphi\colon k \to L$ a morphism of k-algebras. If E/k is a finite, real extension of k, then φ admits a continuation, i.e. a morphism of k-algebras φ' such that the following diagram commutes:*

$$\begin{tikzcd} k \arrow[r, "\varphi"] \arrow[d, hook] & L \\ E \arrow[ur, dashed, "\varphi'"'] & \end{tikzcd} .$$

Proof. Since k is perfect, E/k is separable. Moreover E/k is finite, thus by the primitive element theorem, $E = k[a]$ for $a \in E$. Let $P \in k[t]$ be the minimal polynomial of a over k. Let E^r be an orderable real-closure of E. Thus E^r is a real-closed extension of k that contains a root of P. By 0.7, P has a root $b \in L$. Now define $\psi\colon k[t] \to L$ by $t \mapsto b$ and $\psi|_k = \varphi$. Since b is a root of P, ψ factors through $E = k[a] = k/(P)$ and gives $\varphi'\colon E \to L$. $\square$