0.1 Regular functions

Lemma 0.1. *If $U \subseteq k^n$ is a Zariski-open set and $f_P\colon k^n \to k$ is a polynomial function such that for $x \in U$, $f_P(x) \neq 0$, then the function $\frac{1}{f_P}$ is continuous on U.*

Proof. For all $t \in k$,

$$\left(\frac{1}{f_P}\right)^{-1}(\{t\}) = \left\{x \in U \mid \frac{1}{f_P(x)} = t\right\}$$
$$= \{x \in U \mid t f_P(x) - 1 = 0\}$$
$$= \mathcal{V}(t f_P - 1) \cap U$$

is closed in U. □

Remark 0.2. There can be many continous functions with respect to the Zariski topology. For instance, all bijective maps $f\colon k \to k$ are Zariski-continuous. In algebraic geometry, we will consider only functions which are locally defined by a rational function. We will define them on open subsets of algebrai sets $V \subseteq k^n$, endowed with the topology induced by the Zariski topology of k^n.

Remark 0.3. The open subsets of algebraic sets $V \subseteq k^n$ are exactly the *locally closed subsets* of k^n.

Definition 0.4. Let $X \subseteq k^n$ be a locally closed subset of k^n. A function $f\colon X \to k$ is called *regular at* $x \in X$, if there exist an open subset $x \in U \subseteq X$ and two polynomial functions $P_U, Q_U\colon U \to k$ such that for all $y \in U$, $Q_U(y) \neq 0$ and

$$f(y) = \frac{P_U(y)}{Q_U(y)}.$$

The function $f\colon X \to k$ is called *regular on* X if, for all $x \in X$, f is regular at x.

Example 0.5. A rational fraction $\frac{P}{Q} \in k(T_1, \ldots, T_n)$ defines a regular function on the standard open set $D(Q)$.

Proposition 0.6. *Let $X \subseteq k^n$ be a locally closed subset. If $f\colon X \to k$ is regular, then f is continous.*

Proof. Since continuity is a local property, we may assume $X = \Omega \subseteq k^n$ open and $f = \frac{P}{Q}$ for polynomial functions $P, Q\colon \Omega \to k$ such that $Q(y) \neq 0$. By 0.1 it suffices to prove that if $P, R\colon \Omega \to k$ are continuous, then $PR\colon \Omega \to k$, $z \mapsto P(z)R(z)$ is continuous. Let $t \in k$. Then

$$(PR)^{-1}(\{t\}) = \{z \in \Omega \mid P(z)R(z) - t = 0\}$$
$$= \mathcal{V}(PR - t) \cap \Omega$$

is closed in Ω. □

Remark 0.7. Being a regular function is a local property.

Proposition 0.8. *Let $X \subseteq k^n$ be a locally closed subset of k^n, endowed with the induced topology. The map*

$$\mathcal{O}_X\colon \{\textit{open sets of } X\} \longrightarrow k - \textit{algebras}$$
$$U \longmapsto \{\textit{regular functions on } U\}$$

defines a sheaf of sheaf of k-algebras on X, which is a subsheaf of the sheaf of functions.

Proof. Constants, sums and products of regular functions are regular, thus $\mathcal{O}_X(U)$ is a sub algebra of the k-algebra of functions $U \to k$. Since restricting a function preserves regularity, $\mathcal{O}_X$ is a presheaf. Since being regular is a local property and the presheaf of functions is a sheaf, $\mathcal{O}_X$ is also a sheaf. □

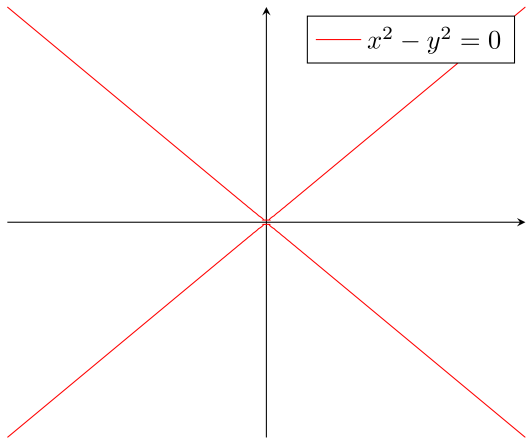

Figure 0.1: Reducible connected algebraic set

0.2 Irreducibility

Definition 0.9. Let X be a topological space. X is

(i) *irreducible* if $X \neq \emptyset$ and X is not the union of two proper closed subets, i.e. for $X = F_1 \cup F_2$ with $F_1, F_2 \subseteq X$ closed, we have $X = F_1$ or $X = F_2$.

(ii) *connected* if X is not the union of two disjoint proper closed subets, i.e. for $X = F_1 \cup F_2$ with $F_1, F_2 \subseteq X$ closed and $F_1 \cap F_2 = \emptyset$, we have $X = F_1$ or $X = F_2$.

A space X which is not irreducible, is called *reducible*.

Lemma 0.10. *If k is infinite, k is irreducible in the Zariski topology.*

Proof. Closed subsets of k are k and finite subsets of k. □

Remark 0.11. If k is finite, k^n is the finite union of its points, which are closed, so k^n is reducible.

Remark 0.12. X irreducible $\implies$ X connected, but the converse is false: Let k be infinite and consider $X = \mathcal{V}_{k^2}(x^2 - y^2)$ (see figure 0.1). Since $x^2 - y^2 = (x-y)(x+y) = 0$ in k if and only if $x = -y$ or $x = y$, we have $X = \mathcal{V}_{k^2}(x-y) \cup \mathcal{V}_{k^2}(x+y)$. Thus X is reducible. But $\mathcal{V}_{k^2}(x-y)$ and $\mathcal{V}_{k^2}(x+y)$ are homeomorphic to k, in particular irreducible and thus connected. Since $\mathcal{V}_{k^2}x - y \cap \mathcal{V}_{k^2}(x+y) \neq \emptyset$, X is connected.

Proposition 0.13. *Let X be a non-empty topological space. The following conditions are equivalent:*

(i) X is irreducible

(ii) If $U_1 \cap U_2 = \emptyset$ with U_1, U_2 open subsets of X, then $U_1 = \emptyset$ or $U_2 = \emptyset$.

(iii) If $U \subseteq X$ is open and non-empty, then U is dense in X.

Proof. Left as an exercise to the reader. □

Proposition 0.14. *Let X be a topological space and $V \subseteq X$. Then V is irreducible if and only if $\overline{V}$ is irreducible.*

Proof. Since $\emptyset$ is closed in X, we have $V = \emptyset \iff \overline{V} = \emptyset$.

($\Rightarrow$) Let $\overline{V} \subseteq Z_1 \cup Z_2$ with $Z_1, Z_2 \subseteq X$ closed. Then $V \subseteq Z_1 \cup Z_2$ and by irreducibility of V we may assume $V \subseteq Z_1$. Since Z_1 is closed, it follows $\overline{V} \subseteq Z_1$.

($\Leftarrow$) Let $V \subseteq Z_1 \cup Z_2$ with $Z_1, Z_2 \subseteq X$ closed. Since $Z_1 \cup Z_2$ is closed, we get $\overline{V} \subseteq Z_1 \cup Z_2$. By irreducibility of $\overline{V}$ we may assume $\overline{V} \subseteq Z_1$, thus $V \subseteq Z_1$. □

Corollary 0.15. *Let X be an irreducible topological space. Then every non-empty open subset $U \subseteq X$ is irreducible.*

Proof. By 0.13, $\overline{U} = X$ and thus irreducible. The claim follows now from 0.14. □

Lemma 0.16 (prime avoidance)**.** *Let $\mathfrak{p}$ be a prime ideal in a commutative ring A. If $I, J \subseteq A$ are ideals such that $IJ \subseteq \mathfrak{p}$, then $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$.*

Proof. Assume that $I \not\subseteq \mathfrak{p}$ and $J \not\subseteq \mathfrak{p}$. Then there exist $a \in I$, such that $a \notin \mathfrak{p}$ and $b \in J$ such that $b \notin \mathfrak{p}$. But $ab \in IJ \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, this implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Contradiction. □

Theorem 0.17. *Let $V \subseteq k^n$ be an algebraic set. Then V is irreducible in the Zariski topology if and only if $\mathcal{I}(V)$ is a prime ideal in $k[T_1, \dots, T_n]$.*

Proof. ($\Rightarrow$) Since $V \neq \emptyset$, $\mathcal{I}(V) \subsetneq k[T_1, \dots, T_n]$. Let $P, Q \in k[T_1, \dots, T_n]$ such that $PQ \in \mathcal{I}(V)$. For $x \in V$, $(PQ)(x) = 0$ in k, hence $P(x) = 0$ or $Q(x) = 0$. Thus $x \in \mathcal{V}(P) \cup \mathcal{V}(Q)$. Therefore $V = (\mathcal{V}(P) \cap V) \cup (\mathcal{V}(Q) \cap V)$ is the union of two closed subsets. Since V is irreducible, we may assume $V = \mathcal{V}(P) \cap V \subseteq \mathcal{V}(P)$, hence $P \in \mathcal{I}(V)$ and $\mathcal{I}(V)$ is prime.

($\Leftarrow$) $V \neq \emptyset$, since $\mathcal{I}(V)$ is a proper ideal. Let $V = V_1 \cup V_2$ with V_1, V_2 closed in V. Then

$$\mathcal{I}(V) = \mathcal{I}(V_1 \cup V_2) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2) \supseteq \mathcal{I}(V_1)\mathcal{I}(V_2).$$

By 0.16, we may assume $\mathcal{I}(V_1) \subseteq \mathcal{I}(V)$. But then

$$V_1 = \mathcal{V}(\mathcal{I}(V_1)) \supseteq \mathcal{V}(\mathcal{I}(V)) = V$$

since V_1 and V are closed. Therefore $V = V_1$ and V is irreducible. □

Corollary 0.18. *If k is infinite, the affine space k^n is irreducible with respect to the Zariski topology.*

Proof. Since k is infinite, $\mathcal{I}(k^n) = (0)$ by **??** which is a prime ideal in the integral domain $k[T_1, \dots, T_n]$. □

Theorem 0.19. *Let $V \subseteq k^n$ be an algebraic set. Then there exists a decomposition*

$$V = V_1 \cup \ldots \cup V_r$$

such that

(i) V_i is a closed irreducible subset of k^n for all i.

(ii) $V_i \not\subseteq V_j$ for all $i \neq j$.

This decomposition is unique up to permutations.

Definition 0.20. For an algebraic set $V \subseteq k^n$, the V_i's in the decomposition in 0.19 are called the *irreducible components* of V.

Proof of 0.19. Existence: Let A be the set of algebraic sets $V \subseteq k^n$ that admit no finite decomposition into a union of closed irreducible subsets. Assume $A \neq \emptyset$. By noetherianity of k^n, there exists a minimal element $V \in A$. In particular V is not irreducible, so $V = V_1 \cup V_2$ with $V_1, V_2 \subsetneq V$. By minimality of V, $V_1, V_2 \notin A$, thus they admit a finite decomposition into a union of closed irreducible subsets. Since $V = V_1 \cup V_2$, the same holds for V. Contradiction. Removing the $V_i's$ for which $V_i \subseteq V_j$ for some j, we may assume that $V_i \not\subseteq V_j$ for $i \neq j$.

Uniqueness: Assume that $V = V_1 \cup \ldots \cup V_r$ and $V = W_1 \cup \ldots \cup W_s$ are decompositions that satisfiy (i) and (ii). Then

$$W_1 = W_1 \cap V = (W_1 \cap V_1) \cup \ldots \cup (W_1 \cap V_r).$$

Since W_1 is irreducible and $W_1 \cap V_i$ is closed in W_1, there exists j such that $W_1 = W_1 \cap V_j \subseteq V_j$. Likewise, there exists k such that $V_j \subseteq W_k$. Hence $W_1 \subseteq W_k$, which forces $k = 1$ (because for $k \neq 1$, we have $W_1 \subsetneq W_k$). Thus $W_1 = V_j$ and we can repeat the procedure with $W_2 \cup \ldots \cup W_s = \bigcup_{i \neq j} V_i$. □

Corollary 0.21. *Let $V \subseteq k^n$ be an algebraic set and denote by $V_1, \ldots, V_r$ the irreducible components of V. Let $W \subseteq V$ be an irreducible subset. Then $W \subseteq V_i$ for some i.*

Proof. We have

$$W = W \cap V = \bigcup_{i=1}^{r} \underbrace{W \cap V_i}_{\text{closed in } W}.$$

Since W is irreducible, there exists an i such that $W = W \cap V_i \subseteq V_i$. □

Remark 0.22. (i) The i in 0.21 is not unique in general. Consider

$$V = \{x^2 - y^2 = 0\} = \{x - y = 0\} \cup \{x + y = 0\}.$$

The closed irreducible subset $\{(0, 0)\}$ lies in the intersection of the irreducible components of V.

(ii) In view of the corollary 0.21, theorem 0.19 implies that an algebraic set $V \subseteq k^n$ has a unique minimal decomposition into a union of closed irreducible subsets.

Corollary 0.23. *Let $V \subseteq k^n$ be an algebraic set. The irreducible components of V are exactly the maximal closed irreducible subsets of V. In terms of ideals in $k[T_1, \ldots, T_n]$, a closed subset $W \subseteq V$ is an irreducible component of V, if and only if the ideal $\mathcal{I}(W)$ is a prime ideal which is minimal among those containing $\mathcal{I}(V)$.*

Proof. A closed irreducible subset $W \subseteq V$ is contained in an irreducible component $V_j \subseteq V$ by 0.21. If W is maximal, then $W = V_j$.

Conversely, if V_j is an irreducible component of V and $V_j \subseteq W$ for some irreducible and closed subset $W \subseteq V$, again by 0.21 we have $W \subseteq V_i$ for some i, therefore $V_j \subseteq V_i$ which implies $i = j$ and $V_j = W$. □

Proposition 0.24 (Identity theorem for regular functions)**.** *Let $X \subseteq k^n$ be an irreducible algebraic set and let $U \subseteq X$ be open. Let $f, g \in \mathcal{O}_X(U)$ be regular functions on U. If there is a non-empty open set $U' \subseteq U$ such that $f|_{U'} = g|_{U'}$, then $f = g$ on U.*

Proof. The set $Y = \mathcal{V}_U(f - g)$ is closed in U and contains U'. Thus the closure $\overline{U'}^{(U)}$ of U' in U is also contained in Y. By 0.15 U is irreducible, so U' is dense in U. Therefore $Y = U$. □

Example 0.25. If k is infinite and $P \in k[T_1, \ldots, T_n]$ is zero outside an algebraic set $V \subsetneq k^n$, then $P = 0$ on k^n.