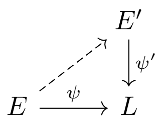
Lemma 0.1. *Let L_1, L_2 be real-closed fields and let $\varphi\colon L_1 \to L_2$ be a homomorphism of fields. Then φ is compatible with the canonical orderings of L_1 and L_2.*

Proof. It suffices to prove that $x \geq_{L_1} 0$ implies $\varphi(x) \geq_{L_2} 0$ for all $x \in L_1$. This follows from the fact that in a real-closed field L, for all $x \in L$, $x \geq 0$ if and only if x is a square. □

Theorem 0.2. *Let $(k, \leq)$ be an ordered field and k^r be a real closure of k that extends the ordering of k. Let L be a orderable real-closed extension of k. Then there exists a unique homomorphism of k-algebras $k^r \to L$.*

Proof. Uniqueness: Let $\varphi\colon k^r \to L$ be a homomorphism of k-algebras and $a \in k^r$. Since a is algebraic over k, it has a minimal polynomial $P \in k[t]$ over k. Denote by $a_1 \leq \ldots \leq a_n$ the roots of P in k^r. Since the characteristic of k is 0, k is perfect, in particular the irreducible polynomial P is separable and thus $a_1 < \ldots < a_n$. Now there exists a unique $1, \ldots, n$ such that $a = a_j$. By **??**, the polynomial P also has n distinct roots $b_1 < \ldots < b_n$ in the real-closed field L. Since φ sends roots of P in k^r to roots of P in L, there is a permutation $\sigma \in S_n$ such that $\varphi(a_i) = b_{\sigma(i)}$. By 0.1, φ respects the ordering of the roots and thus $\sigma = \mathrm{id}$ and $\varphi(a) = \varphi(a_j) = b_j$.

Existence: Consider the set $\mathcal{F}$ of all pairs (E, ψ) where $k \subseteq E \subseteq k^r$ is a subextension of k^r/k and $\psi\colon E \to L$ is a homomorphism of k-algebras. Since $(k, k \hookrightarrow L) \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Define an inductive ordering on $\mathcal{F}$ by $(E, \psi) \leq (E', \psi)$ if there is a commutative diagram



in the category of k-algebras. Then by Zorn, the set $\mathcal{F}$ admits a maximal element (E, ψ). E is real-closed, otherwise it admits a finite real extension E' of E. In particular $E' \subseteq k^r$. Since L is real-closed, $\psi\colon E \to L$ admits a continuation $\psi'\colon E' \to L$ by **??**. Thus $(E, \psi) < (E', \psi')$ contradicting the maximality of (E, ψ). Hence E is real-closed and k^r/E is real algebraic, thus $E = k^r$. So ψ is a homomorphism of k-algebras from k^r to L. □

Corollary 0.3. *Let $(k, \leq)$ be an ordered field. If k_1^r and k_2^r are real closures of k whose canonical orderings are compatible with that of k, then there exists a unique isomorphism of k-algebras $k_1^r \xrightarrow{\simeq} k_2^r$.*

Proof. By 0.2 there exist unique homomorphisms of k-algebras $\varphi\colon k_1^r \to k_2^r$ and $\psi\colon k_2^r \to k_1^r$. Then $\psi \circ \varphi$ and $\mathrm{id}_{k_1^r}$ are homomorphisms $k_1^r \to k_1^r$ of k-algebras. By uniqueness in 0.2, $\psi \circ \varphi = \mathrm{id}_{k_1^r}$. Similarly, $\varphi \circ \psi = \mathrm{id}_{k_2^r}$. □

Remark 0.4. Contrary to the situation of algebraic closures of a field k, for ordered fields $(k, \leq)$ there is a well-defined notion of the real closure of k whose canonical ordering is compatible with that of k. As shown by **??**, it is necessary to fix an ordering of the real field k to get the existence of an isomorphism of fields between two orderable real closures of k.

Corollary 0.5. *Let $(k, \leq)$ be an ordered field and let k^r be the real closure of k. Then k^r has no non-trivial k-automorphism.*

Proof. Take $k_1^r = k_2^r$ in 0.3. □