0.1 The real Nullstellensatz

When k is algebraically closed, Hilbert's Nullstellensatz implies $\mathcal{I}(\mathcal{V}_{k^n}(I)) = \sqrt{I}$ for all ideal $I \subseteq k[T_1, \ldots, T_n]$. In this section we try to compute $\mathcal{I}(\mathcal{V}_{k^n}(I))$ when k is a real-closed field.

Definition 0.1. Let $(k, \leq)$ be an ordered field and let A be a commutative k-algebra with unit. An ideal $I \subseteq A$ is called a *real ideal* if it satisfies the following condition: If $\lambda_1, \ldots, \lambda_r > 0$ in k and $a_1, \ldots, a_r \in A$ satisfy

$$\sum_{j=1}^{r} \lambda_j a_j^2 \in I,$$

then $a_j \in I$ for all j. A is a *real algebra* if the zero ideal in A is a real ideal.

Proposition 0.2. *Let $(k, \leq)$ be an ordered field and let $Z \subseteq k^n$ be a subset. Then the ideal $\mathcal{I}(Z)$ is a real ideal.*

Proof. If $Z = \emptyset$, then $\mathcal{I}(Z) = \mathcal{I}(\emptyset) = k[T_1, \ldots, T_n]$ is a real ideal. Now assume $Z \neq \emptyset$. In this case, if $P_1, \ldots, P_r \in k[T_1, \ldots, T_n]$ and $\lambda_1, \ldots, \lambda_r > 0$ in k are such that $\sum_{j=1}^{r} \lambda_j P_j^2 \in \mathcal{I}(Z)$, then for all $x \in Z$, $\sum_{j=1}^{r} \lambda_j P_j^2(x) = 0$ in k. Since k is an ordered field and $\lambda_j > 0$ for all j, this implies that for all j, $P_j(x) = 0$, i.e. $P_j \in \mathcal{I}(Z)$. $\square$

Recall that if k is an arbitrary field and $I \subsetneq k[T_1, \ldots, T_n]$ is a proper ideal, then finding a common zero $x \in L^n$ to all polynomials $P \in I$ for some extension L of k is equivalent to finding a homomorphism of k-algebras

$$\varphi\colon k[T_1, \ldots, T_n]/I \longrightarrow L.$$

Indeed, the correspondence is obtained by sending such a φ to $x = (x_1, \ldots, x_n)$ where $x_i = \varphi(T_i \bmod I)$. The basic result should be about giving sufficient conditions for such homomorphisms to exist.

Theorem 0.3 (Real Nullstellensatz I). *Let $(k, \leq)$ be an ordered field and let $k^{(r)}$ be the real closure of k. Let $I \subseteq k[T_1, \ldots, T_n]$ be a real ideal. Then there exists a homomorphism of k-algebras*

$$k[T_1, \ldots, T_n]/I \longrightarrow k^{(r)}.$$

In particular, if $I \subsetneq k[T_1, \ldots, T_n]$ is a proper real ideal, then $\mathcal{V}_{k^r}(I) \neq \emptyset$.

Let $(k, \leq)$ be an ordered field. For the proof of 0.3, we need two lemmata:

Lemma 0.4. *Let $I \subseteq k[T_1, \ldots, T_n]$ be a real ideal. Then $\sqrt{I} = I$. Moreover, if $\mathfrak{p} \supset I$ is a minimal prime ideal containing I, then $\mathfrak{p}$ is real.*

Lemma 0.5. *Let $\mathfrak{p} \subseteq k[T_1, \ldots, T_n]$ be a prime ideal. Then the fraction field*

$$K := \mathrm{Frac}\,(k[T_1, \ldots, T_n]/\mathfrak{p})$$

is a real field if and only if the prime ideal $\mathfrak{p}$ is real. In that case K can be ordered in a way that extends the order of k.