0.1 Plane algebraic curves

Theorem 0.1. *If $f \in k[x,y]$ is an irreducible polynomial such that $\mathcal{V}(f)$ is infinite, then $\mathcal{I}(\mathcal{V}(f)) = (f)$. In particular, $\mathcal{V}(f)$ is irreducible in this case.*

Remark 0.2. (i) If k is algebraically closed and $n \geq 2$, then for all $f \in k[x_1, \ldots, x_n]$ non-constant, the zero set $\mathcal{V}(f)$ is necessarily infinite.

(ii) The assumption $\mathcal{V}(f)$ infinite is necessary for the conclusion of 0.1 to hold: The polynomial

$$f(x,y) = (x^2-1)^2 + y^2$$

is irreducible because, as a polynomial in y, it is monic and does not have a root in $\mathbb{R}[x]$ (for otherwise there would be a polynomial $P(x) \in \mathbb{R}[x]$ such that $P(x)^2 = -(x^2-1)^2$) and the zero set of f is

$$\mathcal{V}(f) = \{(1,0)\} \cup \{(-1,0)\},$$

which is reducible.

(iii) 0.1 does not hold in this form for hypersurfaces of k^n for $n \geq 3$. For instance, the polynomial

$$f(x,y,z) = x^2y^2 + z^4 \in \mathbb{R}[x,y,z]$$

is irreducible and the hypersurface

$$\mathcal{V}(f) = \{(0,y,0) \colon y \in \mathbb{R}\} \cup \{(x,0,0) \colon x \in \mathbb{R}\}$$

is infinite. However, the function

$$P \colon (x,y,z) \mapsto xy$$

belongs to $\mathcal{I}(\mathcal{V}(f))$ but not to (f). Moreover, $P \in \mathcal{I}(\mathcal{V}(f))$ but neither x nor y are in $\mathcal{I}(\mathcal{V}(f))$, so this ideal is not prime.

(iv) Take $f(x,y) = (x-a)^2 + y^2 \in \mathbb{R}[x,y]$ which is irreducible. Then $\mathcal{V}(f) = \{(a,0)\}$ is irreducible, and $\mathcal{I}(\mathcal{V}(f)) = (x-a, y) \supsetneq (f)$. In particular, (f) is a non-maximal prime ideal.

We need a special case of the famous Bézout theorem, for which we need a result from algebra. For an integral domain R denote by $Q(R)$ its fraction field. If R is a factorial ring then $q \in R[T]$ is called *primitve* if it is non-constant and its coefficients are coprime in R.

Proposition 0.3 (Gauß). *Let R be a factorial ring. Then $R[T]$ is also factorial. A polynomial $q \in R[T]$ is prime in $R[T]$ if and only if*

(i) $q \in R$ and q is prime in R, or

(ii) q is primitve in $R[T]$ and prime in $Q(R)[T]$

Proof. Any algebra textbook. □

Proposition 0.4. *Let R be a factorial ring and $f, g \in R[X]$ coprime. Then f and g are coprime in $Q(R)[X]$.*

Proof. Let $h = \frac{a}{b} \in Q(R)[X]$ be a common irreducible factor of f and g with $a \in R[X]$ and $b \in R \setminus 0$. By Gauß $R[X]$ is factorial, thus we may assume a irreducible. Then

$$\frac{f}{1} = \frac{p_1}{q_1}\frac{a}{b} \text{ and } \frac{g}{1} = \frac{p_2}{q_2}\frac{a}{b}$$

for some $p_1, p_2 \in R[X]$ and $q_1, q_2 \in R \setminus 0$. So $p_1 a = f q_1 b$ and $p_2 a = g q_2 b$. a neither divides q_1, q_2 nor b, for otherwise $a \in R \setminus 0$ by the degree formula for polynomials and h is a unit. Since a divides $f q_1 b$ and $g q_2 b$ and, since $R[X]$ is factorial, a is prime in $R[X]$ and thus $a \mid f$ and $a \mid g$. □

Lemma 0.5 (Special case of Bézout). *Let $f, g \in k[x,y]$ be two polynomials without common factors in $k[x,y]$. Then the set $\mathcal{V}(f) \cap \mathcal{V}(g)$ is finite.*

Proof. Since $k(x)[y]$ is a principal ideal domain, 0.4 implies $(f,g) = k(x)[y]$, hence the existence of $A(x,y), B(x,y), M(x), N(x)$ such that

$$f(x,y)A(x,y) + g(x,y)B(x,y) = \underbrace{M(x)N(x)}_{=:D(x)}$$

with $D(x) \in k[x]$. Since a common zero (x,y) of f and g gives a zero of D, and D has finitely many zeros, there are only finitely many x such that (x,y) is a zero of both f and g. But, for fixed $x \in k$, the polynomial

$$y \mapsto f(x,y) - g(x,y)$$

has only finitely many zeros in k. So $\mathcal{V}(f) \cap \mathcal{V}(g)$ is finite. □

Proof of 0.1. Let $f \in k[x,y]$ be irreducible such that $\mathcal{V}(f) \subseteq k^2$ is infinite. Since $f \in \mathcal{I}(\mathcal{V}(f))$, it suffices to show that $\mathcal{I}(\mathcal{V}(f)) \subseteq (f)$. Let $g \in \mathcal{I}(\mathcal{V}(f))$. Then $\mathcal{V}(f) \subseteq \mathcal{V}(g)$. Thus

$$\mathcal{V}(f) \cap \mathcal{V}(g) = \mathcal{V}(f)$$

which is infinite by assumption. Thus by 0.5, f and g have a common factor. Since f is irreducible, this implies that $f \mid g$, i.e. $g \in (f)$. □

We can use 0.1 to find the irreducible components of a hypersurface $\mathcal{V}(P) \subseteq k^2$.

Corollary 0.6. *Let $P \in k[x,y]$ be non-constant and $P = uP_1^{n_1} \cdots P_r^{n_r}$ be the decomposition into irreducible factors. If each $\mathcal{V}(P_i)$ is infinite, then the algebraic sets $\mathcal{V}(P_i)$ are the irreducible components of $\mathcal{V}(P)$.*

Proof. Note that

$$\mathcal{V}(P) = \mathcal{V}(P_1^{n_1} \cdots P_r^{n_r}) = \mathcal{V}(P_1) \cup \ldots \cup \mathcal{V}(P_r).$$

Since P_i is irreducible and $\mathcal{V}(P_i)$ is infinite for all i, by 0.1 $\mathcal{V}(P_i)$ is irreducible and for $i \neq j$ $\mathcal{V}(P_i) \not\subset \mathcal{V}(P_j)$, for otherwise

$$(P_i) = \mathcal{I}(\mathcal{V}(P_i)) \supset \mathcal{I}(\mathcal{V}(P_j)) = (P_j)$$

which is impossible for distinct irreducible elements P_i, P_j. □

Example 0.7 (Real plane cubics). Let $P(x,y) = y^2 - f(x)$ with $\deg_x f = 3$ in $k[x]$. Since $\deg_y P \geq 1$ and the leading coefficient of P is 1, the polynomial P is primitive in $k[x][y]$. It is reducible in $k(x)[y]$ if and only if there exists $a(x), b(x) \in k(x)$ such that $(y-a)(y-b) = y^2 - f$, i.e. $b = -a$ and $f = a^2$ in $k(x)$, therefore also in $k[x]$. Since $\deg_x f = 3$, this cannot happen. So, P is irreducible by 0.3.

Moreover, when $k = \mathbb{R}$, the cubic polynomial $f(x)$ takes on an infinite number of positive values, so $\mathcal{V}(y^2 - f(x)) = \mathcal{V}(P)$ is infinite. In conclusion, real cubics of the form $y^2 - f(x) = 0$ are irreducible algebraic sets in $\mathbb{R}^2$ by 0.1.

Proposition 0.8. *Let k be an algebraically closed field and let $P \in k[T_1, \ldots, T_n]$ be a non-constant polynomial with $n \geq 2$. Then $\mathcal{V}(P)$ is infinite.*

Proof. Since P is non-constant, we may assume that $\deg_{x_1} P \geq 1$. Write

$$P(T_1, \ldots, T_n) = \sum_{i=1}^{d} g_i(T_2, \ldots, T_n) T_1^i,$$

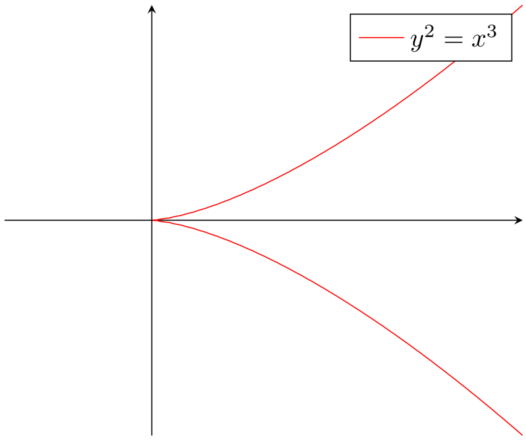

Figure 0.1: the cuspidal cubic

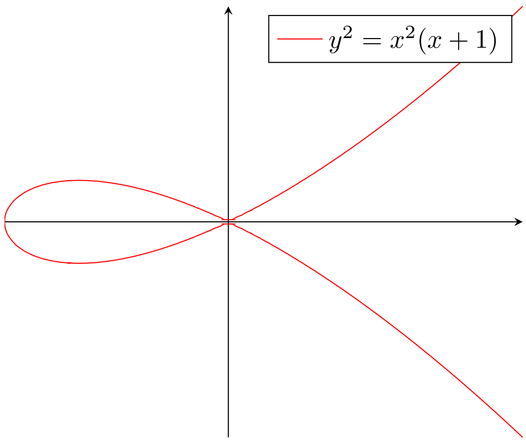

Figure 0.2: the nodal cubic

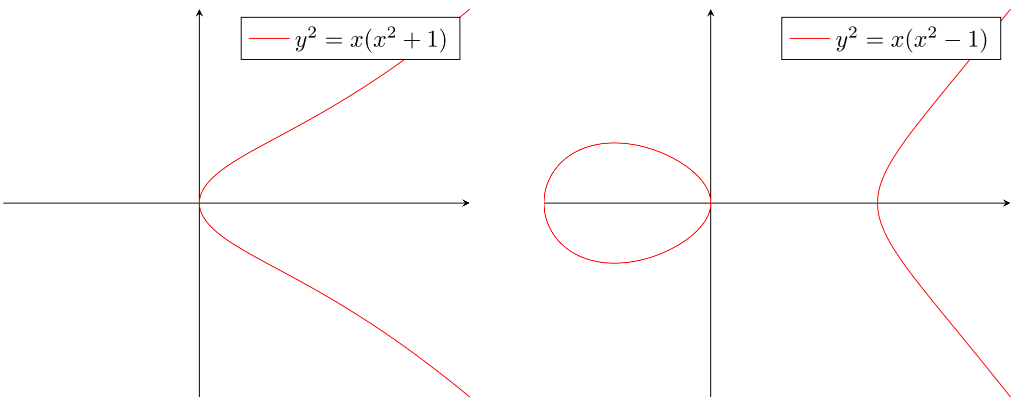

Figure 0.3: the smooth cubics: the second curve demonstrates that the notion of connectedness in the Zariski topologoy of $\mathbb{R}^2$ is very different from the one in the usual topology of $\mathbb{R}^2$.

with $d \geq 1$ and $g_d \neq 0$. Then $D_{k^{n-1}}(g_d)$ is infinite: Since $g_d \neq 0$ and k infinite, it is non-empty. Thus let $(a_2, \ldots, a_n) \in k^{n-1}$ such that $g_d(a) \neq 0$. Then $g_d(ta) = g_d(ta_2, \ldots, ta_n) \in k[t]$ is a non-zero polynomial and thus has only finitely many zeros in k. In particular $D_{k^{n-1}}(g_d)$ is infinite.

For $(a_2, \ldots, a_{n-1}) \in D_{k^{n-1}}(g_d)$, $P(T_1, a_2, \ldots, a_n) \in k[T_1]$ is non-constant and thus has a root a_1 in the algebraically closed field k. Hence $(a_1, \ldots, a_n) \in \mathcal{V}(P)$. □

We finally give a complete classification of irreducible algebraic sets in the affine plane k^2 for an infinite field k.

Proposition 0.9. *Let k be an infinite field. Then the irreducible algebraic subsets of k^2 are:*

(i) the whole affine plane k^2

(ii) single points $\{(a, b)\} \subseteq k^2$

(iii) infinite algebraic sets defined by an irreducible polynomial $f \in k[x, y]$.

Proof. Let $V \subseteq k^2$ be an irreducible algebraic subset of the affine plane. If V is finite, it reduces to a point. So we may assume V infinite. If $\mathcal{I}(V) = (0)$, then $V = k^2$. Otherwise, there is a non-constant polynomial $P \in k[x, y]$ such that P vanishes on V. Since V is irreducible, $\mathcal{I}(V)$ is prime, so it contains an irreducible factor f of P. Let $g \in \mathcal{I}(V)$. Then $V \subseteq \mathcal{V}(f) \cap \mathcal{V}(g)$, but since V is infinite, f and g must have a common factor. By irreducibility of f, it follows $f \mid g$, i.e. $g \in (f)$. Hence $\mathcal{I}(V) = (f)$ and $V = \mathcal{V}(f)$. □