0.1 Abstract affine varieties

Recall that an isomorphism of spaces with functions is a morphism $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ that admits an inverse morphism.

Remark 0.1. As we have seen, a bijective morphism is not necessarily an isomorphism.

Remark 0.2. Somewhat more formally, one could also define a morphism of spaces with functions (over k) to be a pair (f, φ) such that $f\colon X \to Y$ is a continuous map and $\varphi\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is the morphism of sheaves f^*. The question then arises how to define properly the composition $(g, \psi) \circ (f, \varphi)$. The formal answer is $(g \circ f, f_*(\varphi) \circ \psi)$.

Definition 0.3. Let k be a field. An (abstract) *affine variety over k* (also called an affine k-variety) is a space with functions $(X, \mathcal{O}_X)$ over k that is isomorphic to the space with functions $(V, \mathcal{O}_V)$, where V is an algebraic subset of some affine space k^n and $\mathcal{O}_V$ is the sheaf of regular functions on V.

A morphism of affine k-varieties is a morphism of the underlying spaces with functions.

Example 0.4. (i) An algebraic subset $V \subseteq k^n$, endowed with its sheaf of regular functions $\mathcal{O}_V$, is an affine variety.

(ii) It is perhaps not obvious at first, but a standard open set $D_V(f)$, where $f\colon V \to k$ is a regular function on an algebraic set $V \subseteq k^n$, defines an affine variety. Indeed, when equipped with its sheaf of regular functions, $D_V(f) \simeq \mathcal{V}_{k^{n+1}}(tf(x) - 1)$.

Remark 0.5. Let $(X, \mathcal{O}_X)$ be a space with functions. An open subset $U \subseteq X$ defines a space with functions $(U, \mathcal{O}_U)$. If $(U, \mathcal{O}_U)$ is isomorphic to some standard open set $D_V(f)$ of an algebraic set $V \subseteq k^n$, we will call U an *affine open set*.

Then the observation is the following: since an algebraic set $V \subseteq k^n$ is a finite union of standard open sets, every point x in an affine variety X has an affine open neighbourhood.

Less formally, an affine variety X, locally „looks like“ a standard open set $D_V(f) \subseteq k^n$, where $V \subseteq k^n$ is an algebraic set. In particular, open subsets of an affine variety also locally look like standard open sets. In fact, they are finite unions of such sets.

Example 0.6. The algebraic group $\mathrm{GL}(n; k)$ is an affine variety over k.

Remark 0.7. An algebraic set $(V, \mathcal{O}_V)$ is a subset $V \subseteq k^n$ defined by polynomial equations and equipped with its sheaf of regular functions. An affine variety $(X, \mathcal{O}_X)$ is „like an algebraic set“ but without a reference to a particular „embedding“ in affine space. This is similar to having a finitely generated k-Algebra A without specifying a particular isomorphism

$$A \simeq k[X_1, \dots, X_n]/I.$$

The next example will illustrate precisely this fact.

Example 0.8. Let us now give an abstract example of an affine variety. We consider a finitely generated k-algebra A and define $X := \mathrm{Hom}_{k-\mathrm{Alg}}(A, k)$. The idea is to think of X as points on which we can evaluate elements of A, which are thought of as functions on X. For $x \in \mathrm{Hom}_k(A, k)$ and $f \in A$ we set $f(x) := x(f) \in k$.

- Topology on X: for all ideal $I \subseteq A$, set

$$\mathcal{V}_X(I) := \{x \in X \mid \forall x \in I\colon f(x) = 0\}.$$

These subsets of X are the closed sets of a topology on X, which we may call the Zariski topology.

- Regular functions on X: if $U \subseteq X$ is open, a function $h\colon U \to k$ is called regular at $x \in U$ if there it exists an open set $x \in U_x$ and elements $P, Q \in A$ such that for $y \in U_x$, $Q(y) \neq 0$ and $h(y) = \frac{P(y)}{Q(y)}$ in k.

  The function h is called regular on U iff it is regular at $x \in U$. Regular functions then form a sheaf of k-algebras on X.

  Moreover, if $h\colon U \to k$ is regular on X, the set $D_X(h) := \{x \in X \mid h(x) \neq 0\}$ is open in X and the function $\frac{1}{h}$ is regular on $D_X(h)$.

So, we have defined a space with functions $(X, \mathcal{O}_X)$, at least whenever $X \neq \emptyset$. We show that X is an affine variety.

Proof. Fix a system of generators of A, i.e.

$$A \simeq k[t_1, \dots, t_n]/I$$

where $k[t_1, \dots, t_n]$ is a polynomial algebra. We denote by $\overline{t_1}, \dots, \overline{t_n}$ the images of $t_1, \dots, t_n$ in A and we define

$$\varphi\colon X = \mathrm{Hom}_k(A, k) \to k^n$$
$$x \mapsto (x(\overline{t_1}), \dots, x(\overline{t_n})).$$

Let $P \in I$ and $x \in X$. Then

$$P(\varphi(x)) = P(x(\overline{t_1}), \dots, x(\overline{t_n})) = x(\overline{P}) = 0.$$

Thus $\varphi(x) \in \mathcal{V}_{k^n}(I)$. Conversely let $a = (a_1, \dots, a_n) \in \mathcal{V}_{k^n}(I)$, then we can define a morphism of k-algebras

$$x_a\colon A \to A/(\overline{t_1} - a_1, \dots, \overline{t_n} - a_n) \simeq k$$

which satisfies $x_a(\overline{t_i}) = a_i$ for all i. So $(a_1, \dots, a_n) = \varphi(x_a) \in \mathrm{im}\ \varphi$.

In particular, we have defined a map

$$\psi\colon \mathcal{V}_{k^n}(I) \to X = \mathrm{Hom}_k(A, k)$$
$$a \mapsto x_a$$

such that $\varphi \circ \psi = \mathrm{Id}_{\mathcal{V}_{k^n}(I)}$. In fact, we also have $\psi \circ \varphi = \mathrm{Id}_X$.

It remains to check that φ and ψ are morphisms of spaces with functions, which follows from the definition of the topology and the notion of regular function on X. □

The elements of $X := \mathrm{Hom}_k(A, k)$ are also called the *characters* of the k-algebra A, and this is sometimes denoted by $\hat{A} := \mathrm{Hom}_{k-\mathrm{alg}}(A, k)$. Note that $\hat{A}$ is a k-subalgebra of the algebra of all functions $f\colon A \to k$.

The character x_a introduced above and associated to an alemenet $a \in A$ is then denoted by $\hat{a}$ and called the *Gelfand transform* of a. The *Gelfand transformation* is the morphism of k-algebras

$$A \to \hat{A}$$
$$a \mapsto \hat{a}.$$

Exercise 1. Let A be a finitely generated k-algebra and let $X = \mathrm{Hom}_{k\text{-alg}}(A, k)$. Show that the map $x \mapsto \ker x$ induces a bijection

$$X \simeq \{\mathfrak{m} \in \mathrm{Spm}\, A \mid A/\mathfrak{m} \simeq k\}.$$

Remark 0.9. Note that we have not assumed A to be reduced and that, if we set $A_{\mathrm{red}} := A/\sqrt{(0)}$, then A_{red} is reduced and $\hat{A_{\mathrm{red}}} = \hat{A}$, because a maximal ideal of A necessarily contains $\sqrt{(0)}$ and the quotient field is „the same“.

Remark 0.10. Let $(X, \mathcal{O}_X)$ be an affine variety. One can associate the k-algebra $\mathcal{O}_X(X)$ of globally defined regular functions on X:

$$\mathcal{O}_X(X) = \{f \colon X \to k \mid f \text{ regular on } X\}.$$

Moreover, if $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism between two affine varieties, we have a k-algebra homomorphism

$$\begin{aligned} \varphi^* \colon \mathcal{O}_Y(Y) &\to \mathcal{O}_X(X) \\ f &\mapsto f \circ \varphi. \end{aligned}$$

Also, $(\mathrm{id}_X)^* = \mathrm{id}_{\mathcal{O}_X(X)}$ and $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ whenever $\psi\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ is a morphism of affine varieties. In other words, we have defined a (contravariant) functor k-Aff $\to$ k-Alg.

Proposition 0.11. *Let k be a field. The functor*

$$\begin{aligned} k\text{-}\mathit{Aff} &\to k\text{-}\mathit{Alg} \\ (X, \mathcal{O}_X) &\mapsto \mathcal{O}_X(X) \end{aligned}$$

is fully faithful.

Proof. Since X and Y are affine, we may assume $X = V \subseteq k^n$ and $Y = W \subseteq k^m$. Then $\varphi\colon V \to W$ is given by m regular functions $(\varphi_1, \dots, \varphi_m)$ on V. On k^m, let us denote by y_i the projection to the i-th factor. Its restriction to W is a regular function

$$y_i|_W \colon W \to k$$

that satisfies $\varphi^*(y_i|_W) = \varphi_i$.

Since for all regular functions $f\colon W \to k$ one has

$$\varphi^* f = f \circ \varphi = f(\varphi_1, \dots, \varphi_m),$$

we see that the morphism

$$\varphi^* \colon \mathcal{O}_W(W) \to \mathcal{O}_V(V)$$

is entirely determined by the m regular functions $\varphi^*(y_i|_W) = \varphi_i$ on V. In particular, if $\varphi^* = \psi^*$, then $\varphi_i = \varphi^*(y_i|_W) = \psi^*(y_i|_W) = \psi_i$, so $\varphi = \psi$, which proves that $\varphi \mapsto \varphi^*$ is injective.

Surjectivity: Let $h\colon \mathcal{O}_W(W) \to \mathcal{O}_V(V)$ be a morphism of k-algebras. Let

$$\varphi \coloneqq (h(y_1|_W), \dots, h(y_m|_W))$$

which is a morphism from V to k^m, because its components are regular functions on V. It satisfies $\varphi^*(y_i|_W) = \varphi_i = h(y_i|_W)$, so $\varphi^* = h$.

It remains to show, that $\varphi(V) \subseteq W$. Let $W = \mathcal{V}(P_1, \dots, P_r)$ with $P_j \in k[Y_1, \dots, Y_m]$. Then for all $j \in \{1, \dots, r\}$ and $x \in V$

$$P_j(\varphi(x)) = P_j(h(y_1|_W), \dots, h(y_m|_W))(x).$$

Since h is a morphism of k-algebras and P_j is a polynomial, we have

$$P_j(h(y_1|_W), \dots, h(y_m|_W)) = h(P_j(y_1|_W), \dots, P_j(y_m|_W)).$$

But $P_j \in \mathcal{I}(W)$, so

$$P_j(y_1|_W, \dots, y_m|_W) = P_j(y_1, \dots, y_m)|_W = 0,$$

which proves that for $x \in V$, $\varphi(x) \in W$. □