

0.1 Geometric Noether normalisation

Consider a plane algebraic curve \mathcal{C} , defined by the equation $f(x, y) = 0$. If we fix $x = a$, then the polynomial equation $f(a, y) = 0$ has only finitely many solutions (at most $\deg_y f$). This means that the map

$$\mathcal{C} := \mathcal{V}(f) \rightarrow k(x, y) \mapsto x$$

has finite fibres. A priori, such a map is not surjective, e.g. for $f(x, y) = xy - 1$. If k is algebraically closed, one can always find such a surjective projection.

Theorem 0.1. *Let k be an algebraically closed field and $f \in k[x_1, \dots, x_n]$ be a polynomial of degree $d \geq 1$. Then there is a morphism of affine varieties*

$$\pi: \mathcal{V}_{k^n}(f) \rightarrow k^{n-1}$$

such that:

(i) π is surjective

(ii) for $t \in k^{n-1}$, the fibre $\pi^{-1}(\{t\}) \subseteq \mathcal{V}(f)$ consists of at most d points.

Proof. Let $f \in k[x_1, \dots, x_n]$ be of degree d . We construct a change of variables of the form $(x_i \mapsto x_i + a_i x_n)_{1 \leq i \leq n-1}$ and $x_n \mapsto x_n$, such that the term of degree d of $f(x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1} x_n, x_n)$ becomes $c x_n^d$ with $c \in k^\times$. Since

$$f(x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1} x_n, x_n) = \sum_{(i_1, \dots, i_n) \in \mathbb{N}^n} \alpha_{i_1, \dots, i_n} (x_1 + a_1 x_n)^{i_1} \cdots (x_{n-1} + a_{n-1} x_n)^{i_{n-1}} x_n^{i_n},$$

the coefficient of x_n^d in the above equation is obtained by considering all (i_1, \dots, i_n) such that $i_1 + \dots + i_n = d$, and keeping only the term in $x_n^{i_j}$ when expanding $(x_j + a_j x_n)^{i_j}$, so we get

$$\sum_{(i_1, \dots, i_n) \in \mathbb{N}^{i_1 + \dots + i_n = d}} \alpha_{i_1, \dots, i_n} a_1^{i_1} \cdots a_{n-1}^{i_{n-1}},$$

which is equal to $f_d(a_1, \dots, a_{n-1}, 1)$, where f_d is the (homogeneous) degree d part of f .

Claim: There exist $a_1, \dots, a_{n-1} \in k$ such that $f_d(a_1, \dots, a_{n-1}, 1) \neq 0$. Proof of claim by induction: if $n = 1$, $f_d = c x_1^d$ for some $c \neq 0$, so $f_d(1) = c \neq 0$. If $n \geq 2$, we can write

$$f_d(x_1, \dots, x_n) = \sum_{i=0}^d h_i(x_2, \dots, x_n) x_1^i$$

where $h_i \in k[x_2, \dots, x_n]$ is homogeneous of degree $d - i$. Since $f_d \neq 0$, there is at least one i_0 such that $h_{i_0} \neq 0$. By induction, we can find $(a_2, \dots, a_{n-1}) \in k^{n-2}$ such that $h_{i_0}(a_2, \dots, a_{n-1}, 1) \neq 0$. But then $f(\cdot, a_2, \dots, a_{n-1}, 1) \in k[x_1]$ is a non zero polynomial, so it has only finitely many roots. As k is infinite, there exists $a_1 \in k$, such that $f(a_1, \dots, a_{n-1}, 1) \neq 0$.

Then

$$\varphi: \begin{cases} x_i \mapsto x_i + a_i x_n & 1 \leq i \leq n-1 \\ x_n \mapsto x_n \end{cases}$$

is an invertible linear transformation $k^n \rightarrow k^n$, such that

$$(f \circ \varphi^{-1})(y_1, \dots, y_n) = c(y_n^d + g_1(y_1, \dots, y_n) y_n^{d-1} + \dots + g_d(y_1, \dots, y_{n-1}))$$

for $c \neq 0$. This induces an isomorphism of affine varieties

$$\begin{aligned} \mathcal{V}(f) &\rightarrow \mathcal{V}(f \circ \varphi^{-1}) \\ x &\mapsto \varphi(x) \end{aligned}$$

such that

$$\begin{array}{ccc} \mathcal{V}(f) & \xrightarrow{\varphi} & k^n = k^{n-1} \times k \\ & \searrow \pi & \downarrow \\ & & k^{n-1} \end{array}$$

defines the morphism π with the desired properties. Indeed: Let $(x_1, \dots, x_n) \in k^n$ and set $y_i := \varphi(x_i)$. Then

$(x_1, \dots, x_n) \in \mathcal{V}(f)$ iff $x_n = y_n$ is a root of the polynomial

$$t^d + \sum_{j=1}^d g_j(y_1, \dots, y_{n-1})t^{d-j}.$$

Therefore for all $t = (y_1, \dots, y_{n-1}) \in k^{n-1}$, $\pi^{-1}(\{t\}) \neq \emptyset$ (because $\bar{k} = k$) and $\pi^{-1}(\{t\})$ has at most d points. \square

Definition 0.2. Let $f \in k[x_1, \dots, x_n]$ be a polynomial of degree d . As in the proof of 0.1, there exists a linear coordinate transformation $\varphi: k^n \rightarrow k^n$, such that $f \circ \varphi^{-1}(y_1, \dots, y_n) = cy_n^d + \sum_{j=1}^d g_j(y_1, \dots, y_{n-1})y_n^{d-j}$. For a point $x \in \pi^{-1}(y_1, \dots, y_{n-1}) \subseteq \mathcal{V}(f)$, the *multiplicity* of x is the multiplicity of y_n as a root of that polynomial.

A point with multiplicity ≥ 2 are called *ramification point* and its image lies in the *discriminant locus* of π .

With this vocabulary, we can refine the statement of 0.1.

Definition 0.3 (Geometric Noether normalisation). Assume $k = \bar{k}$. If $f \in k[x_1, \dots, x_n]$ is polynomial of degree d , a morphism of affine varieties

$$\pi: \mathcal{V}_{k^n}(f) \rightarrow k^{n-1}$$

such that

- (i) π is surjective
- (ii) for $t \in k^{n-1}$, the number of elements in $\pi^{-1}(\{t\})$, counted with their respective multiplicities, is exactly d ,

is called a *geometric Noether normalisation*.

Corollary 0.4 (Geometric Noether normalisation for hypersurfaces). *Let k be an algebraically closed field and $f \in k[x_1, \dots, x_n]$ be a polynomial of degree $d \geq 1$. Then there exists a geometric Noether normalisation.*

Example 0.5. Let $f(x, y) = y^2 - x^3 \in \mathbb{C}[x, y]$. Then the map

$$\mathcal{V}_{\mathbb{C}^2}(y^2 - x^3) \rightarrow \mathbb{C}(x, y) \quad \mapsto y$$

is a geometric Noether normalisation, but $(x, y) \mapsto x$ is not (the fibres of the latter have degree 2, while $\deg f = 3$).

Remark 0.6. In the proof of 0.1, to construct φ and the g_j , we only used that k is infinite. Thus the statement, that for all $f \in k[x_1, \dots, x_n]$ there exists a linear automorphism $\varphi: k^n \rightarrow k^n$ such that

$$f \circ \varphi^{-1}(y_1, \dots, y_n) = c \left(y_n^d + \sum_{j=1}^d g_j(y_1, \dots, y_{n-1})y_n^{d-j} \right)$$

is valid over k if k is infinite. The resulting map

$$\pi: \mathcal{V}_{k^n}(f) \rightarrow k^{n-1}$$

still has finite fibres, but it is no longer surjective in general, as the example $f(x, y) = x^2 + y^2 - 1$ shows.

However, it induces a surjective map with finite fibres

$$\hat{\pi}: \mathcal{V}_{\bar{k}^n}(f) \rightarrow \bar{k}^{n-1}$$

which moreover commutes with the action of $\text{Gal}(\bar{k}/k)$.

Theorem 0.7. *Let k be an infinite field and \bar{k} an algebraic closure of k . Let $f \in k[x_1, \dots, x_n]$ be a polynomial of degree $d \geq 1$. Then there exists a $\text{Gal}(\bar{k}/k)$ -equivariant geometric Noether normalisation map $\pi: \mathcal{V}_{\bar{k}^n}(f) \rightarrow \bar{k}^{n-1}$.*

Example 0.8. Let $f(x, y) = y^2 - x^3 \in \mathbb{R}[x, y]$. Then the map

$$\begin{aligned} \pi: \mathcal{V}_{\mathbb{C}^2}(y^2 - x^3) &\rightarrow \mathbb{C} \\ (x, y) &\mapsto y. \end{aligned}$$

is a geometric Noether normalisation map and it is Galois-invariant:

$$\pi(\overline{(x, y)}) = \pi(\bar{x}, \bar{y}) = \bar{y} = \overline{\pi(x, y)}.$$

Exercise 1. Show that if $y \in \mathbb{R}$, the group $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\pi^{-1}(\{y\})$, and that the fixed point set of that action is in bijection with $\{x \in \mathbb{R} \mid y^2 - x^3 = 0\}$.

Next, we want to generalise the results above beyond the case of hypersurfaces.

Theorem 0.9. *Assume k is algebraically closed. Let $V \subseteq k^n$ be an algebraic set. Then there exists a natural number $r \leq n$ and a morphism of algebraic sets*

$$p: V \rightarrow k^r$$

such that p is surjective and has finite fibres.

Sketch of proof. If $V = k^n$, we take $r = n$ and $p = \text{id}_{k^n}$. Otherwise $V = \mathcal{V}(I)$ with $I \subseteq k[x_1, \dots, x_n]$ a non-zero ideal. Take $f \in I \setminus \{0\}$. Then there exists a geometric Noether normalisation

$$p_1: \mathcal{V}(f) \rightarrow k^{n-1}.$$

One can now show that $V_1 := p_1(V)$ is an algebraic set in k^{n-1} . Thus there are two cases:

- (1) $p_1(V) = k^{n-1}$. Thus $p_1|_V: V \rightarrow k^{n-1}$ is surjective with finite fibres and we are done.
- (2) $p_1(V) \subsetneq k^{n-1}$. In this case $p_1(V) = \mathcal{V}(I_1)$ with $I_1 \subseteq k[x_1, \dots, x_{n-1}]$ a non-zero ideal. So we can repeat the argument.

After $r \leq n$ steps, the above algorithm terminates, and this happens precisely when $V_r = k^{n-r}$. If we set

$$p := p_r \circ \dots \circ p_1: V \rightarrow k^{n-r}$$

then p is surjective with finite fibres because $p(V) = V_r = k^{n-r}$ and each p_i has finite fibres. \square

Remark 0.10. By the fact used in the proof of 0.9, p is in fact a closed map. Note that when $r = n$, $V = p^{-1}(\{0\})$ is actually finite, in which case $\dim V$ should indeed be 0.