0.1 Gluing spaces with functions

We present a general technique to construct spaces with functions by „patching together“ other spaces with functions „along open subsets“. This will later be used to argue that, in order to define a structure of variety on a topological sapce (or even a set), it suffices to give one atlas.

Theorem 0.1 (Gluing theorem). *Let $(X_i, \mathcal{O}_{X_i})_{i \in I}$ be a family of spaces with functions. For all pair (i,j), assume that the following has been given*

(a) *an open subset $X_{ij} \subseteq X_i$*

(b) *an isomorphism of spaces with functions*

$$\varphi_{ji} \colon (X_{ij}, \mathcal{O}_{X_{ij}}) \to (X_{ji}, \mathcal{O}_{X_{ji}})$$

subject to the following compatibility conditions

(1) *for all i, $X_{ii} = X_i$ and $\varphi_{ii} = id_{X_i}$*

(2) *for all pair (i,j), $\varphi_{ij} = \varphi_{ji}^{-1}$*

(3) *for all triple (i,j,k), $\varphi_{ji}(X_{ik} \cap X_{ij}) = X_{jk} \cap X_{ji}$ and $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ on $X_{ik} \cap X_{ij}$.*

Then there exists a space with functions $(X, \mathcal{O}_X)$ equipped with a family of open sets $(U_i)_{i \in I}$ and isomorphisms of spaces with functions

(A1) $\varphi_i \colon (U_i, \mathcal{O}_X|_{U_i}) \to (X_i, \mathcal{O}_{X_i})$,

such that $\bigcup_{i \in I} U_i = X$ and, for all pair (i,j),

(B2) $\varphi_i(U_i \cap U_j) = X_{ij}$, *and*

(C3) $\varphi_j \circ \varphi_i^{-1} = \varphi_{ji}$ *on X_{ij}.*

Such a familiy $(U_i, \varphi_i)_{i \in I}$ is called an atlas for $(X, \mathcal{O}_X)$.

Moreover, if $(Y, \mathcal{O}_Y)$ is a space with functions equipped with an atlas $(V_i, \psi_i)_{i \in I}$ satisfying conditions (A1), (A2) and (A3), then the isomorphisms $\psi_i^{-1} \circ \varphi_i \colon U_i \to V_i$ induce an isomorphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

Proof. Uniqueness up to canonical isomorphism: Let $(U_i, \varphi_i)_{i \in I}$ and $(V_i, \psi_i)_{i \in I}$ be two atlases modelled on the same gluing data, then for all pair (i,j),

$$\psi_j^{-1} \circ \varphi_j \Big|_{U_i \cap U_j} = \psi_j^{-1} \circ \underbrace{(\varphi_j \circ \varphi_i^{-1})}_{=\varphi_{ji}} \circ \varphi_i \Big|_{U_i \cap U_j}$$

$$= \psi_j^{-1} \circ \underbrace{(\psi_j \circ \psi_i^{-1})}_{=\varphi_{ji}} \circ \varphi_i \Big|_{U_i \cap U_j}$$

$$= \psi_i^{-1} \circ \varphi_i \Big|_{U_i \cap U_j}$$

so there is a well-defined map

$$f \colon X = \bigcup_{i \in I} U_i \to \bigcup_{i \in I} V_i = Y$$

$$(x \in U_i) \mapsto (\psi_i^{-1} \circ \varphi_i(x) \in V_i)$$

which induces an isomorphism of spaces with functions.

Existence: Define $\tilde{X} := \bigsqcup_{i \in I} X_i$ and let the topology be the final topology with respect to the canonical maps $(X_i \to \tilde{X})_{i \in I}$. Then define $X := \tilde{X}/\sim$ where $(i,x) \sim (j,y)$ in $\tilde{X}$ if $x = \varphi_{ij}(y)$.

Conditions (1), (2) and (3) show that $\sim$ is reflexive, symmetric and transitive. We equip X with the quotient topology and denote by

$$p \colon \tilde{X} \to X$$

the canonical continuous projection. Let $U_i := p(X_i)$. Since $p^{-1}(U_i) = \bigsqcup_{j \in I} X_{ji}$ is open in $\tilde{X}$, U_i is open in X. Moreover, $\bigcup_{i \in I} U_i = X$, so we have an open covering of X. We put $p_i := p|_{X_i}$ and we define a sheaf on X by setting

$$\mathcal{O}_X(U) := \{f \colon U \to k \mid \forall i \in I, f \circ p_i \in \mathcal{O}_{X_i}(p_i^{-1}(U))\}$$

for all open sets $U \subseteq X$. This defines a sheaf on X, with respect to which $(X, \mathcal{O}_X)$ is a space with functions. Finally, $p_i \colon X_i \to U_i$ is a homeomorphism and, by construction $\mathcal{O}_{U_i} \simeq (p_i)_* \mathcal{O}_{X_i}$ via pullback by p_i. We have thus constructed a space with functions $(X, \mathcal{O}_X)$, equipped with an open covering $(U_i)_{i \in I}$ and local charts

$$\varphi_i := p_i^{-1} \colon (U_i, \mathcal{O}_X|_{U_i}) \overset{\sim}{\longrightarrow} (X_i, \mathcal{O}_{X_i}).$$

It remains to check that $\varphi_i(U_i \cap U_j) = X_{ij}$ and $\varphi_j \circ \varphi_i^{-1} = \varphi_{ji}$ on X_{ij}, but this follows from the construction of $X = \bigsqcup_{i \in I} X_i / \sim$ and the definition of the φ_i's as $p|_{X_i}^{-1}$. $\qquad \square$

Example 0.2. Take $k = \mathbb{R}$ or $\mathbb{C}$ equipped with either the Zariski or the usual topology. Consider the spaces with functions $X_1 = k$, $X_2 = k$ and the open sets $X_{12} = k \setminus \{0\} \subseteq X_1$ and $X_{21} = k \setminus \{0\} \subseteq X_2$. Finally, set

$$\varphi_{21} \colon X_{12} \to X_{21}$$
$$t \mapsto \frac{1}{t}.$$

Since this is an isomorphism of spaces with functions, we can glue X_1 and X_2 along $X_{12} \xrightarrow[\varphi_{21}]{\sim} X_{21}$ and define a space with functions $(X, \mathcal{O}_X)$ with an atlas modelled on (X_1, X_2, φ_{21}). We will now identify this space X with the projective line $k\mathbb{P}^1$. By definition, the latter is the set of 1-dimensional vector subspaces (lines) of k^2:

$$k\mathbb{P}^1 := (k^2 \setminus \{0\})/k^\times.$$

Then, we have a covering $U_1 \cup U_2 = k\mathbb{P}^1$, where $U_1 = \{[x_1 : x_2] \mid x_1 \neq 0\}$ and $U_2 = \{[x_1 : x_2] \mid x_2 \neq 0\}$, and we can define charts

$$\varphi_1 \colon U_1 \overset{\sim}{\longrightarrow} k$$
$$[x_1 : x_2] \longmapsto x_2/x_1$$
$$[1 : w] \longleftarrow w$$

and $\varphi_2 \colon U_2 \to k$ likewise. Then, on the intersection

$$U_1 \cap U_2 = \{[x_1 : x_2] \mid x_1 \neq 0, x_2 \neq 0\}$$

we have a commutative diagram

$$\begin{array}{ccc} U_1 \cap U_2 & & \\ \Big\downarrow{\varphi_1} & \searrow{\varphi_2} & \\ X_1 & \xrightarrow{\varphi_{21}} & X_2 \end{array}$$

with $\varphi_i(U_1 \cap U_2)$ open in X_i. In view of the gluing theorem, we can use this to set up a bijection $k\mathbb{P}^1 \to X$ where $X := (X_1 \sqcup X_2)/\sim_{\varphi_{12}}$ and define a topology and a sheaf of regular functions on $k\mathbb{P}^1$ via this identification. Note that this was done without putting a topology on $k\mathbb{P}^1$: the latter is obtained using the bijection $k\mathbb{P}^1 \to X$ constructed above. We now spell out the notion of regular functions thus obtained on $k\mathbb{P}^1$.

Proposition 0.3. *With the identification*

$$k\mathbb{P}^1 = X_1 \sqcup X_2/\sim$$

constructed above, a function $f \colon U \to k$ defined on an open subset $U \subseteq k\mathbb{P}^1$ is an element of $\mathcal{O}_X(U)$ if and only if, for each local chart $\varphi_i \colon U_i \to k$, the function

$$f \circ \varphi_i^{-1} \colon \varphi_i(U_i \cap U) \to k$$

is regular on the open set $\varphi_i(U_i \cap U) \subseteq k$.

Definition 0.4. Let k be a field. An *algebraic k-prevariety* is a space with functions $(X, \mathcal{O}_X)$ such that

(i) X is quasi-compact.

(ii) $(X, \mathcal{O}_X)$ is locally isomorphic to an affine variety.

Remark 0.5. Saying that $(X, \mathcal{O}_X)$ is locally isomorphic to an affine variety means that for $x \in X$, it exists an open neighbourhood $x \in U$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an open subset of an affine variety. Since such an open set is a union of principal open sets, which are themselves affine, one can equivalently ask that $(U, \mathcal{O}_U)$ be affine. Thus:

Proposition 0.6. *A space with functions $(X, \mathcal{O}_X)$ is an algebraic prevariety, if and only if there exists a finite open covering*

$$X = U_1 \cup \ldots \cup U_n$$

such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine variety.

Remark 0.7. As a consequence of the gluing theorem, in order to either construct an algebraic prevariety or put a structure of an algebraic prevariety on a set, it suffices to either define X from certain gluing data $(X_i, X_{ij}, \varphi_{ij})_{(i,j)}$ satisfying appropriate compatibility conditions, or find a covering $(U_i)_{i \in I}$ of a set X and local charts $\varphi_i \colon U_i \to X_i$ such that $X_{ij} = \varphi_i(U_i \cap U_j)$ is open in X_i and $\varphi_j \circ \varphi_i^{-1}$ is an isomorphism of spaces with functions.

In practice, X is sometimes given as a topological space, and $(U_i)_{i \in I}$ is an open covering, with local charts $\varphi_i \colon U_i \to X_i$ that are homeomorphisms. So the condition that X_{ij} be open in X_i is automatic in this case and one just has to check that

$$\varphi_j \circ \varphi_i^{-1} \colon X_{ij} \to X_{ji}$$

induces an isomorphism of spaces with functions. In the present context where X_i and X_j are affine varieties, this means a map

$$X_{ij} \subseteq k^n \to X_{ji} \subseteq k^m$$

between locally closed subsets of k^n and k^m whose components are regular functions.

Example 0.8 (Projective sets). We have already seen that projective spaces $k\mathbb{P}^n$ are algebraic pre-varieties. Let $P \in k[x_0, \ldots, x_n]_d$ be a homogeneous polynomial of degree $d \geq 0$. Although P cannot be evaluated at a point $[x_0 : \ldots : x_n] \in k\mathbb{P}^n$, the condition $P(x_0, \ldots, x_n) = 0$ can be tested, because for $\lambda \in k^x$,

$$P(x_0, \ldots, x_n) = 0 \iff 0 = \lambda^d P(x_0, \ldots, x_n) = P(\lambda x_0, \ldots, \lambda x_n).$$

We use this to define the following *projective sets*

$$\mathcal{V}_{k\mathbb{P}^n}(P_1, \ldots, P_m) = \{[x_0 : \ldots : x_n] \in k\mathbb{P}^n \mid P_i(x_0, \ldots, x_n) = 0 \quad \forall i\}$$

for homogeneous polynomials in $(x_0, \ldots, x_n)$.

We claim that these projective sets are the clsoed sets of a topology on $k\mathbb{P}^n$, called the Zariski topology. A basis for that topology is provided by the principal open sets $D_{k\mathbb{P}^n}(P)$ where P is a homogeneous polynomial. By definition, a regular function on a locally closed subset of $k\mathbb{P}^n$ is locally given by the restriction of a ration fraction of the form

$$\frac{P(x_0,\ldots,x_n)}{Q(x_0,\ldots,x_n)}$$

where P and Q are homogeneous polynomials of the same degree. This defines a sheaf of regular functions on any given locally closed subset X of $k\mathbb{P}^n$.

Proposition 0.9. *A Zariski-closed subset X of $k\mathbb{P}^n$ equipped with its sheaf of regular functions, is an algebraic pre-variety. The same holds for all open subsets $U \subseteq X$.*

Proof. Consider the open covering

$$X = \bigcup_{i=0}^{n} X \cap U_i$$
$$= \bigcup_{i=0}^{n} \{[x_0 : \ldots : x_n] \in X \mid x_i \neq 0\}.$$

Then the restriction to $X \cap U_i$ of the local chart

$$\varphi_i \colon U_i \longrightarrow k^n$$
$$x = [x_0 : \ldots : x_n] \longmapsto \underbrace{\left(\frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}\right)}_{w=(w_0,\ldots,\hat{w}_i,\ldots,w_n)}$$

sends an x such that $P_1(x) = \ldots = P_m(x) = 0$ to a w such that $Q_1(w) = \ldots = Q_m(w) = 0$ where, for all j,

$$Q_j(w) = P_j(w_0,\ldots,w_{i-1},1,w_{i+1},\ldots,w_n)$$
$$= P_j(x_0,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)$$

is the dehomogeneisation of P_j. So $\varphi_i(X \cap U_i) = \mathcal{V}_{k^n}(Q_1,\ldots,Q_m) =: X_i$ is an algebraic subset of k^n, in particular an affine variety. It remains to check that $\varphi_i|_{X \cap U_i}$ pulls back regular functions on X_i to regular functions on $X \cap U_i$, and similarly for $(\varphi_i|_{X \cap U_i})^{-1}$. But if f and g are polynomials in $(w_0,\ldots,\hat{w}_i,\ldots,w_n)$,

$$\left(\varphi_i^* \frac{f}{g}\right)(x) = \frac{f(\varphi_i(x))}{g(\varphi_i(x))}$$
$$= \frac{f\left(\frac{x_0}{x_i},\ldots,\frac{\hat{x}_i}{x_i},\ldots,\frac{x_n}{x_i}\right)}{g\left(\frac{x_0}{x_i},\ldots,\frac{\hat{x}_i}{x_i},\ldots,\frac{x_n}{x_i}\right)}$$

which can be rewritten as a quotient of two homogeneous polynomials of the same degree by multiplying the numerator and denominator by x_i^r with $r \geq \max(\deg(f),\deg(g))$. The computation is similar but easier for $(\varphi_i|_{X \cap U_i})^{-1}$. $\square$

Definition 0.10. A space with functions $(X, \mathcal{O}_X)$ which is isomorphic to a Zariski-closed subset of $k\mathbb{P}^n$ is called a *projective k-variety*.