

Lemma 0.1. *The category of affine varieties admits products.*

Proof. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be affine varieties. Choose embeddings $X \subseteq k^n$ and $Y \subseteq k^p$ for some n and p . Then $X \times Y \subseteq k^{n+p}$ is an affine variety, endowed with two morphisms of affine varieties $\text{pr}_1: X \times Y \rightarrow X$ and $\text{pr}_2: X \times Y \rightarrow Y$. We will prove that the triple $(X \times Y, \text{pr}_1, \text{pr}_2)$ satisfies the universal property of the product of X and Y .

Let $f_X: Z \rightarrow X$ and $f_Y: Z \rightarrow Y$ be morphisms of affine varieties. Then define $f = (f_X, f_Y): Z \rightarrow X \times Y$. This satisfies $\text{pr}_1 \circ f = f_X$ and $\text{pr}_2 \circ f = f_Y$. If we embed Z into some k^m , the components of f_X and f_Y are regular functions from k^m to k^n and k^p . Thus the components of $f = (f_X, f_Y)$ are regular functions $k^m \rightarrow k^{n+p}$, i.e. f is a morphism. \square

Theorem 0.2. *The category of algebraic pre-varieties admits products.*

Proof. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ algebraic pre-varieties. Let

$$X = \bigcup_{i=1}^r X_i \text{ and } Y = \bigcup_{j=1}^s Y_j$$

be affine open covers. Then, as a set,

$$X \times Y = \bigcup_{i,j} X_i \times Y_j.$$

By 0.1, each $X_i \times Y_j$ has a well-defined structure of affine variety. Moreover, if $X'_i \subseteq X_i$ and $Y'_j \subseteq Y_j$ are open sets, then $X'_i \times Y'_j$ is open in $X_i \times Y_j$.

So we can use the identity morphism to glue $X_{i_1} \times Y_{j_1}$ to $X_{i_2} \times Y_{j_2}$ along the common open subset $(X_{i_1} \cap X_{i_2}) \times (Y_{j_1} \cap Y_{j_2})$. This defines an algebraic prevariety P whose underlying set is $X \times Y$. Also, the canonical projections $X_i \times Y_j \rightarrow X_i$ and $X_i \times Y_j \rightarrow Y_j$ glue together to give morphisms $p_X: X \times Y \rightarrow X$ and $p_Y: X \times Y \rightarrow Y$, which coincide with pr_1 and pr_2 .

There only remains to prove the universal property. Let $f_X: Z \rightarrow X$ and $f_Y: Z \rightarrow Y$ be morphisms of algebraic prevarieties and set $f = (f_X, f_Y): Z \rightarrow X \times Y$. In particular, $\text{pr}_1 \circ f = f_X$ and $\text{pr}_2 \circ f = f_Y$ as maps between sets. To prove that f is a morphism of algebraic prevarieties, it suffices to show that this is locally the case. Z is covered by the open subsets $f_X^{-1}(X_i) \cap f_Y^{-1}(Y_j)$, each of which can be covered by affine open subsets $(W_l^{ij})_{1 \leq l \leq q(i,j)}$. By construction, $f(W_l^{ij}) \subseteq X_i \times Y_j$. So, by the universal property of the affine variety $X_i \times Y_j$, the map $f|_{W_l^{ij}}$ is a morphism of affine varieties. \square

Definition 0.3 (algebraic variety). Let (X, \mathcal{O}_X) be an algebraic pre-variety and $X \times X$ the product in the category of algebraic pre-varieties. If the subset

$$\Delta_X := \{(x, y) \in X \times X \mid x = y\}$$

is closed in $X \times X$, then (X, \mathcal{O}_X) is said to be an *algebraic variety*. A morphism of algebraic varieties $f: X \rightarrow Y$ is a morphism of the underlying pre-varieties.

Example 0.4 (of a non-separated algebraic prevariety). We glue two copies X_1, X_2 of k along the open subsets $k \setminus \{0\}$ using the isomorphism of spaces with functions $t \mapsto t$. The resulting algebraic prevariety is a „line with two origins”, denoted by 0_1 and 0_2 . For this prevariety X , the diagonal Δ_X is not closed in $X \times X$.

Indeed, if Δ_X were closed in $X \times X$, then its pre-image in $X_1 \times X_2$ under the morphism

$f: X_1 \times X_2 \rightarrow X \times X$ defined by

$$\begin{array}{ccc}
 X_1 \times X_2 & \xrightarrow{\quad i_1 \circ \text{pr}_1 \quad} & X \\
 \downarrow i_2 \circ \text{pr}_2 & \searrow \text{dashed} & \downarrow \\
 & X \times X & \longrightarrow X \\
 & & \downarrow \\
 & & X
 \end{array}$$

where $i_j: X_j \hookrightarrow X$ is the canonical inclusion of X_j into $X = (X_1 \sqcup X_2) / \sim$, would be closed in $X_1 \times X_2$. But

$$\begin{aligned}
 f^{-1}(\Delta_X) &= \{(x_1, x_2) \in X_1 \times X_2 \mid i_1(x_1) = i_2(x_2)\} \\
 &= \{(x_1, x_2) \in X_1 \times X_2 \mid x_j \neq 0 \text{ and } x_1 = x_2 \text{ in } k\} \\
 &= \{(x, x) \in k \times k \mid x \neq 0\} \subseteq k \times k = X_1 \times X_2
 \end{aligned}$$

which is not closed in $X_1 \times X_2$. In fact, $f^{-1}(\Delta_X) = \Delta_k \setminus \{(0, 0)\} \subseteq k \times k$.

Corollary 0.5. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be algebraic varieties, then the product in the category of algebraic pre-varieties is an algebraic variety. In particular the category of algebraic varieties admits products.*

Proof. $\Delta_{X \times Y} \simeq \Delta_X \times \Delta_Y \subseteq (X \times X) \times (Y \times Y)$. □

Proposition 0.6. *Affine varieties are algebraic varieties.*

Proof. Let X be an affine variety. We choose an embedding $X \subseteq k^n$. Then $\Delta_X = \Delta_{k^n} \cap (X \times X)$. But

$$\Delta_{k^n} = \{(x_i, y_i)_{1 \leq i \leq n} \in k^{2n} \mid x_i - y_i = 0\}$$

is closed in k^{2n} . Therefore, Δ_X is closed in $X \times X$ (note that the prevariety topology of $X \times X$ coincides with its induced topology as a subset of k^{2n} by construction of the product prevariety $X \times X$). □

Exercise 0.7. Let (X, \mathcal{O}_X) be an algebraic pre-variety and let $Y \subseteq X$ be a closed subset. For all open subsets $U \subseteq Y$, we set

$$\mathcal{O}_Y(U) := \left\{ h: U \rightarrow k \mid \forall x \in U \exists \hat{U} \subseteq X \text{ open, } \hat{U} \cap U = \{x\}, g \in \mathcal{O}_X(\hat{U}) \text{ such that } g|_{\hat{U} \cap U} = h|_{\hat{U} \cap U} \right\}.$$

- Show that this defines a sheaf of regular functions on Y and that (Y, \mathcal{O}_Y) is an algebraic prevariety.
- Show that the canonical inclusion $i_Y: Y \hookrightarrow X$ is a morphism of algebraic prevarieties and that if $f: Z \rightarrow X$ is a morphism of algebraic prevarieties such that $f(Z) \subseteq Y$, then f induces a morphism $\tilde{f}: Y \rightarrow Z$ such that $i_Y \circ \tilde{f} = f$.
- Show that, if X is an algebraic variety, then Y is also an algebraic variety.

Recall that $k\mathbb{P}^n$ is the projectivisation of the k -vector space k^{n+1} :

$$k\mathbb{P}^n = P(k^{n+1})(k^{n+1} \setminus \{0\})/k^\times.$$

Proposition 0.8 (Segre embedding). *The k -bilinear map*

$$\begin{aligned}
 k^{n+1} \times k^{m+1} &\longrightarrow k^{n+1} \otimes_k k^{m+1} \simeq k^{(n+1)(m+1)} \\
 (x, y) &\longmapsto x \otimes y
 \end{aligned}$$

induces an isomorphism of algebraic pre-varieties

$$P(k^{n+1}) \times P(k^{m+1}) \xrightarrow{f} \zeta \subseteq P\left(k^{(n+1)(m+1)}\right) = k\mathbb{P}^{nm+n+m}$$

$$([x_0 : \dots : x_n], [y_0 : \dots : y_m]) \longmapsto [x_0y_0 : \dots : x_0y_m : \dots : x_ny_0 : \dots : x_ny_m]$$

where ζ is a Zariski-closed subset of $k\mathbb{P}^{nm+n+m}$.

Proof. It is clear that f is well-defined. Let us denote by $(z_{ij})_{0 \leq i \leq n, 0 \leq j \leq m}$ the homogeneous coordinates on $k\mathbb{P}^{nm+n+m}$, and call them *Segre coordinates*. Then $f(k\mathbb{P}^n \times k\mathbb{P}^m)$ is contained in the projective variety

$$\zeta = \mathcal{V}(\{z_{ij}z_{kl} - z_{kj}z_{il} \mid 0 \leq i, k \leq n, 0 \leq j, l \leq m\})$$

$$\subseteq P\left(k^{(n+1)(m+1)}\right)$$

as can be seen by writing

$$f([x], [y]) = \begin{bmatrix} x_0y_0 & \dots & x_0y_m \\ \vdots & & \vdots \\ x_ny_0 & \dots & x_ny_m \end{bmatrix}$$

so that

$$z_{ij}z_{kl} - z_{kj}z_{il} = \begin{vmatrix} x_iy_j & x_iy_l \\ x_ky_j & x_ky_l \end{vmatrix} = 0.$$

The map f is injective because, if $z := f([x], [y]) = f([x'], [y'])$ then there exists (i, j) such that $z \in W_{ij} := \{z \in k\mathbb{P}^{nm+n+m} \mid z_{ij} \neq 0\}$ so $x_iy_j = x'_iy'_j \neq 0$. In particular $\frac{x_j}{x_i} = \frac{y'_j}{y_j} = \lambda \neq 0$. Since

$$[x_0y_0 : \dots : x_ny_m] = [x'_0y'_0 : \dots : x'_ny'_m]$$

means that there exists $\mu \neq 0$ such that, for all (k, l) , $x_ky_l = \mu x'_ky'_l$. Taking $k = i$ and $l = j$, we get that $\mu = 1$ and hence, for all k , $x_ky_j = x'_ky'_j$, so $x_k = \frac{y'_j}{y_j} x'_k = \lambda x'_k$. Likewise, for all l , $x_iy_l = x'_iy'_l$, so $y_l = \frac{1}{\lambda} y'_l$. As a consequence $[x_0 : \dots : x_n] = [x'_0 : \dots : x'_n]$ and $[y_0 : \dots : y_m] = [y'_0 : \dots : y'_m]$, thus proving that f is injective. Note that we have proven that

$$f^{-1}(W_{ij}) = U_i \times V_j$$

where $U_i = \{[x] \in k\mathbb{P}^n \mid x_i \neq 0\}$ and $V_j = \{[y] \in k\mathbb{P}^m \mid y_j \neq 0\}$.

For simplicity, let us assume that $i = j = 0$. The open sets U_0, V_0, W_0 are affine charts, in which f is equivalent to

$$k^n \times k^m \longrightarrow k^{nm+n+m}$$

$$(u, v) \longmapsto (v_1, \dots, v_m, u_1, u_1v_1, \dots, u_1v_m, \dots, u_n, u_nv_1, \dots, v_nv_m)$$

which is clearly regular. In particular $f|_{U_0 \times V_0}$ is a morphism of algebraic pre-varieties.

im $f = \zeta$: Let $[z] \in \zeta$. Since the W_{ij} cover $k\mathbb{P}^{nm+n+m}$, we can assume without loss of generality, $z_{00} \neq 0$. Then by definition of ζ , $z_{kl} = \frac{z_{k0}z_{0l}}{z_{00}}$ for all (k, l) . If we set

$$([x_0 : \dots : x_n], [y_0 : \dots : y_m]) = \left(\left[1 : \frac{z_{10}}{z_{00}} : \dots : \frac{z_{n0}}{z_{00}} \right], \left[1 : \frac{z_{01}}{z_{00}} : \dots : \frac{z_{0m}}{z_{00}} \right] \right)$$

we have a well defined point $([x], [y]) \in U_0 \times V_0 \subseteq k\mathbb{P}^n \times k\mathbb{P}^m$, which satisfies $f([x], [y]) = [z]$.

Thus $f^{-1} : \zeta \rightarrow k\mathbb{P}^n \times k\mathbb{P}^m$ is defined and a morphism of algebraic pre-varieties because, in affine charts $W_0 \xrightarrow{f^{-1}|_{W_0}} U_0 \times V_0$ as above, it is the regular map $(u_{ij})_{(i,j)} \mapsto ((u_{i0})_i, (u_{0j})_j)$. \square

Corollary 0.9. *Projective varieties are algebraic varieties.*

Proof. By 0.7 it suffices to show that $k\mathbb{P}^n$ is an algebraic variety. Let $f: k\mathbb{P}^n \times k\mathbb{P}^n \rightarrow k\mathbb{P}^{n^2+2n}$ be the Segre embedding. For $[x] \in k\mathbb{P}^n$:

$$f([x], [x]) = \begin{bmatrix} x_0x_0 & \dots & x_0x_m \\ \vdots & & \vdots \\ x_nx_0 & \dots & x_nx_m \end{bmatrix}.$$

Thus $f([x], [x])_{ij} = f([x], [x])_{ji}$. Let now $[z] \in \zeta \subseteq k\mathbb{P}^{n^2+2n}$, where ζ is defined in the proof of 0.8, and such that, in Segre coordinates, $z_{ij} = z_{ji}$. Without loss of generality, we can assume $z_{00} = 1$. Set $x_j := z_{0j}$ for $1 \leq j \leq n$. Thus for all (i, j)

$$f([x], [y])_{ij} = x_ix_j = z_{0i}z_{0j} = z_{i0}z_{0j} = z_{ij}z_{00} = z_{ij},$$

i.e.

$$\Delta_{k\mathbb{P}^n} \simeq \{[z] \in \zeta \mid z_{ij} = z_{ji}\}$$

which is a projective and thus closed set of $k\mathbb{P}^n \times k\mathbb{P}^n$. □