

## 0.1 Real closures

**Proposition 0.1.** *Let  $k$  be a real field. Then there exists a real-closed algebraic orderable extension  $k^r$  of  $k$ .*

*Proof.* Let  $\bar{k}$  be an algebraic closure of  $k$  and  $E$  be the set of intermediate extensions  $k \subseteq L \subseteq \bar{k}$  such that  $L$  is real and algebraic over  $k$ .  $E \neq \emptyset$  since  $k \in E$ . Define  $L_1 < L_2$  on  $E$  if and only if  $L_1 \subseteq L_2$  and  $L_2/L_1$  is ordered, i.e. the order relation on  $L_1$  coincides with the one induced by  $L_2$ . Then every totally ordered family  $(E_i)_{i \in I}$  has an upper bound, namely  $\bigcup_{i \in I} E_i$ . By Zorn,  $E$  has a maximal element, which we denote by  $k^r$  and which is an algebraic extension of  $k$ . Such a  $k^r$  is real-closed, because otherwise it would admit a proper real algebraic extension contradicting the maximality of  $k^r$  as a real algebraic extension of  $k$ .  $\square$

**Definition 0.2.** A real-closed real algebraic extension of a real field  $k$  is called a *real closure* of  $k$ .

**Remark 0.3.** By the construction in the proof of 0.1, a real closure of a real field  $k$  can be chosen as a subfield  $k^r$  of an algebraic closure of  $\bar{k}$ . Since  $k^r[i]$  is algebraically closed and algebraic over  $k^r$ , so also over  $k$ , it follows  $k^r[i] = \bar{k}$ .

**Proposition 0.4.** *Let  $k$  be a real field and  $L$  be a real-closed extension of  $k$ . Let  $\bar{k}^L$  be the relative algebraic closure of  $k$  in  $L$ , i.e.*

$$\bar{k}^L = \{x \in L \mid x \text{ algebraic over } k\}.$$

*Then  $\bar{k}^L$  is a real closure of  $k$ .*

*Proof.* It is immediate that  $\bar{k}^L$  is a real algebraic extension of  $k$ . Let  $x \in \bar{k}^L$ . Then  $x$  or  $-x$  is a square in  $L$ , since  $L$  is real-closed. Without loss of generality, assume that  $x \in L^{[2]}$ . Then  $t^2 - x \in \bar{k}^L[t]$  has a root in  $L$ . Since this root is algebraic over  $\bar{k}^L$ , hence over  $k$ , it belongs to  $\bar{k}^L$ . Thus  $x$  is in fact a square in  $\bar{k}^L$ . By the same argument every polynomial of odd degree has a root in  $\bar{k}^L$ .  $\square$

**Example 0.5.** (i)  $\bar{\mathbb{Q}}^{\mathbb{R}} = \bar{\mathbb{Q}}^{\mathbb{C}} \cap \mathbb{R}$  is a real closure of  $\mathbb{Q}$ . In particular,  $\bar{\mathbb{Q}}^{\mathbb{C}} = \bar{\mathbb{Q}}^{\mathbb{R}}[i]$  as subfields of  $\mathbb{C}$ .

(ii) Consider the real field  $k = \mathbb{R}(t)$  and the real-closed extension

$$\widehat{\mathbb{R}(t)} = \bigcup_{q>0} \mathbb{R}((t^{1/q})).$$

Then the subfield  $\overline{\widehat{\mathbb{R}(t)}}^{\mathbb{R}(t)}$ , consisting of all those real Puiseux series that are algebraic over  $\mathbb{R}(t)$ , is a real closure of  $\mathbb{R}(t)$ .

The field of real Puiseux series itself is a real closure of the field  $\mathbb{R}((t))$  of real formal Laurent series.

Real-closed fields  $L$  admit a canonical structure of ordered field, where  $x \geq 0$  in  $L$ , if and only if  $x$  is a square. In particular, if  $k$  is a real field and  $k^r$  is a real closure of  $k$ , then  $k$  inherits an ordering from  $k^r$ . However, different real closures may induce different orderings on  $k$ , as the next example shows.

**Example 0.6.** Let  $k = \mathbb{Q}(t)$ . This is a real field, since  $\mathbb{Q}$  is real. Since  $\pi$  is transcendental over  $\mathbb{Q}$ , we can embed  $\mathbb{Q}(t)$  in  $\mathbb{R}$  by sending  $t$  to  $\pi$ .

$$i_1: \mathbb{Q}(t) \xrightarrow{\sim} \mathbb{Q}(\pi) \subseteq \mathbb{R}.$$

Since  $\mathbb{R}$  is real-closed, the relative algebraic closure  $i_1(\mathbb{Q}(t))^{\mathbb{R}}$  is a real closure of  $i_1(\mathbb{Q}(t))$ .

We can also embed  $\mathbb{Q}(t)$  in the field  $\widehat{\mathbb{R}(t)}$  of real Puiseux series via a homomorphism  $i_2$  and then  $\overline{i_2(\mathbb{Q}(t))^{\widehat{\mathbb{R}(t)}}}$  is a real closure of  $i_2(\mathbb{Q}(t))$ . However, the ordering on  $\overline{i_1(\mathbb{Q}(t))^{\mathbb{R}}}$  is Archimedean, because it is a subfield of  $\mathbb{R}$ , while the ordering on  $\overline{i_2(\mathbb{Q}(t))^{\widehat{\mathbb{R}(t)}}}$  is not Archimedean (it contains infinitesimal elements, such as  $t$  for instance).

The fields  $\overline{i_1(\mathbb{Q}(t))^{\mathbb{R}}}$  and  $\overline{i_2(\mathbb{Q}(t))^{\widehat{\mathbb{R}(t)}}}$  cannot be isomorphic as fields. Indeed, when two real-closed fields  $L_1, L_2$  are isomorphic as fields, then they are isomorphic as ordered fields, since positivity on a real closed field is defined by the condition of being a square, which is preserved under isomorphisms of fields.

The next result will be proved later on.

**Lemma 0.7.** *Let  $(k, \leq)$  be an ordered field and  $P \in k[t]$  be an irreducible polynomial. Let  $L_1, L_2$  be real-closed extensions of  $k$  that are compatible with the ordering of  $k$ . Then  $P$  has the same number of roots in  $L_1$  as in  $L_2$ .*