0.1 Prime ideals in $k[x,y]$

Proposition 0.1. *Let A be a principal ideal domain. Let $\mathfrak{p} \subseteq A[X]$ be a prime ideal. Then $\mathfrak{p}$ satisfies exactly one of the following three mutually exclusive possibilities:*

(i) $\mathfrak{p} = (0)$

(ii) $\mathfrak{p} = (f)$, *where $f \in A[X]$ is irreducible*

(iii) $\mathfrak{p} = (a, q)$, *where $a \in A$ is irreducible and $q \in A[X]$ such that its reduction modulo aA is an irreducible element in $A/aA[X]$. In this case, $\mathfrak{p}$ is a maximal ideal.*

Proof. Let $\mathfrak{p} \subseteq A[X]$ be a prime ideal. If $\mathfrak{p}$ is principal, then $\mathfrak{p} = (f)$ for some $f \in A[X]$. If $f = 0$, we are done. Otherwise, since $A[X]$ is factorial by Gauß and $\mathfrak{p}$ is prime, f is irreducible.

Let now $\mathfrak{p}$ not be principal. Then there exist $f, g \in \mathfrak{p}$ without common factors in $A[X]$. By **??**, they also have no common factors in the principal ideal domain $Q(A)[X]$, so $Mf + Ng = 1$ for some $M, N \in Q(A)[X]$. By multiplying with the denominators, we obtain $Pf + Qg = b$ for some $b \in A$ and $P, Q \in A[X]$. So $b \in (f, g) \subseteq \mathfrak{p}$, thus there is an irreducible factor a of b in A such that $a \in \mathfrak{p}$. Moreover, $aA[X] \subsetneq \mathfrak{p}$ since $\mathfrak{p}$ is not principal. Now consider the prime ideal

$$\mathfrak{p}/aA[X] \subset A[X]/aA[X] \simeq (A/aA)[X].$$

Since A is a PID and a is irreducible, A/aA is a field and $(A/aA)[X]$ a PID. So $\mathfrak{p}/aA[X]$ is generated by an irreducible element $\overline{q} \in (A/aA)[X]$ for some $q \in A[X]$. Thus $\mathfrak{p} = (a, q)$. Moreover

$$A[X]/\mathfrak{p} \simeq \left(A/aA\right)[X] \big/ \left(\mathfrak{p}/aA\right)[X] = \left(A/aA\right)[X] \big/ \overline{q}\left(A/aA\right)[X]$$

which is a field since $\left(A/aA\right)[X]$ is a PID. So $\mathfrak{p}$ is maximal in $A[X]$. □

Using 0.1 we can give a simple proof for the classification of maximal ideals of $k[T_1, \ldots, T_n]$ when k is algebraically closed and $n = 2$.

Corollary 0.2. *If k is algebraically closed, a maximal ideal $\mathfrak{m}$ of $k[x,y]$ is of the form $\mathfrak{m} = (x-a, y-b)$ with $(a,b) \in k^2$. In particular, principal ideals are never maximal.*

Proof. Since $\mathfrak{m}$ is maximal, it is prime and $\mathfrak{m} \neq (0)$. By 0.1, $\mathfrak{m} = (P, f)$ with $P \in k[x]$ irreducible and $f \in k[x,y]$ such that its image $\overline{f}$ in $(k[x]/(P))[y]$ is irreducible or $\mathfrak{m} = (f)$ for $f \in k[x,y]$ irreducible.

(1) $\mathfrak{m} = (P, f)$. Since k is algebraically closed and $P \in k[x]$ is irreducible, $P = x - a$ for some $a \in k$.

$$k[x]/(P) = k[x]/(x-a) \simeq k.$$

Since $\overline{f} \in k[y]$ is also irreducible, $\overline{f} = y - b$ for some $b \in k$.

(2) $\mathfrak{m} = (f)$. Since $k = \overline{k}$, $\mathcal{V}(f)$ is infinite, in particular $\mathcal{V}(f) \neq \emptyset$. Then if $(a,b) \in \mathcal{V}(f)$,

$$(x-a, y-b) = \mathcal{I}(\{(a,b)\}) \supset \mathcal{I}(\mathcal{V}(f)) \supset (f).$$

Since (f) is maximal, it follows that $(f) = (x-a, y-b)$, which is impossible since $x - a$ and $y - b$ habe no common factors in $k[x,y]$.

□

Remark 0.3. The ideal (x^2+1, y) is maximal in $\mathbb{R}[x,y]$ and is not of the form $(x-a, y-b)$ for $(a,b) \in \mathbb{R}^2$. Indeed,

$$\mathbb{R}[x,y]\big/(x^2+1, y) \simeq \left(\mathbb{R}[y]/y\mathbb{R}[y]\right)[x]\big/(x^2+1) \simeq \mathbb{R}[x]/(x^2+1) \simeq \mathbb{C}.$$

Proposition 0.4. *Let k be an algebraically closed field. Then the maps $V \mapsto \mathcal{I}(V)$ and $I \mapsto \mathcal{V}(I)$ induce a bijection*

$$\{\textit{irreducible algebraic subsets of } k^2\} \longleftrightarrow \{\textit{prime ideals in } k[x,y]\}$$

through wich we have correspondences

$$\begin{aligned} \textit{points } (a,b) \in k^2 &\longleftrightarrow \textit{maximal ideals } (x-a, y-b) \textit{ in } k[x,y] \\ \textit{proper, infinite, irreducible algebraic sets} &\longleftrightarrow \textit{prime ideals } (f) \subseteq k[x,y] \textit{ with } f \textit{ irreducible} \\ k^2 &\longleftrightarrow (0). \end{aligned}$$

Proof. Let $V \subseteq k^2$ be an irreducible algebraic set. By **??** we can distinguish the following cases:

(i) If $V = k^2$, then $\mathcal{I}(V) = (0)$ since k is infinite and $\mathcal{I}(\mathcal{V}(0)) = (0)$.

(ii) If $V = \{(a,b)\}$, then $\mathcal{I}(V) \supset (x-a, y-b) =: \mathfrak{m}$. Since $\mathfrak{m}$ is maximal, $\mathcal{I}(V) = \mathfrak{m}$. Since $V = \mathcal{V}(\mathfrak{m})$, this also shows $\mathcal{I}(\mathcal{V}(\mathfrak{m})) = \mathfrak{m}$.

(iii) If $V = \mathcal{V}(f)$ where $f \in k[x,y]$ is irreducible, then by **??** $\mathcal{I}(\mathcal{V}(f)) = (f)$.

So, every irreducible algebraic set $V \subseteq k^2$ is of the form $\mathcal{V}(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subseteq k[x,y]$. Moreover,

$$\mathcal{I}(\mathcal{V}(\mathfrak{p})) = \mathfrak{p}.$$

Let now $\mathfrak{p}$ be a prime ideal in $k[x,y]$. By 0.1 we can dinstiguish the following cases:

(i) $\mathfrak{p} = (0)$: Then $\mathcal{V}(\mathfrak{p}) = k^2$ and since k is infinite, k^2 is irreducible.

(ii) $\mathfrak{p}$ maximal: Then by 0.2, $\mathfrak{p} = (x-a, y-b)$ for some $(a,b) \in k^2$. So $\mathcal{V}(m) = \{(a,b)\}$ is irreducible.

(iii) $\mathfrak{p} = (f)$ with $f \in k[x,y]$ irreducible. Since $k = \overline{k}$, $\mathcal{V}(f)$ is infinite and hence by **??** irreducible.

Thus the maps in the proposition are well-defined, mutually inverse and induce the stated correspondences. □

Corollary 0.5. *Assume that k is algebraically closed and let $\mathfrak{p} \subseteq k[x,y]$ be a prime ideal. Then*

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \text{ maximal}, \mathfrak{m} \supset \mathfrak{p}} \mathfrak{m}.$$

Proof. If $\mathfrak{p}$ is maximal, there is nothing to prove. If $\mathfrak{p} = (0)$, $\mathfrak{p}$ is contained in $(x-a, y-b)$ for $(a,b) \in k^2$. Since k is infinite, the intersection of these ideals is (0). Otherwise, by 0.4, $\mathfrak{p} = (f)$ for some $f \in k[x,y]$ irreducible. Then, since $k = \overline{k}$, $\mathcal{V}(f)$ is infinite and with **??**:

$$\mathfrak{p} = (f) = \mathcal{I}(\mathcal{V}(f)) = \mathcal{I}\left(\bigcup_{(a,b) \in \mathcal{V}(f)} \{(a,b)\} \right) \supset \bigcap_{(a,b) \in \mathcal{V}(f)} \mathcal{I}(\{(a,b)\}) \supset (f) = \mathfrak{p}.$$

By 0.4, the ideals $\mathcal{I}(\{(a,b)\})$ for $(a,b) \in \mathcal{V}(f)$ are exactly the maximal ideals containing $(f) = \mathfrak{p}$. □

Corollary 0.6. *Let $\mathfrak{p} \subseteq k[x,y]$ be a non-principal prime ideal. Then $\mathcal{V}(\mathfrak{p}) \subseteq k^2$ is finite.*

Proof. Since $\mathfrak{p}$ is not principal, there exist $f, g \in \mathfrak{p}$ without common factors. Since $(f,g) \subset \mathfrak{p}$, we have

$$\mathcal{V}(f) \cap \mathcal{V}(g) = \mathcal{V}(f,g) \supset \mathcal{V}(\mathfrak{p})$$

and the left hand side is finite by **??**. □