0.1 The tangent cone and the Zariski tangent space

0.1.1 The tangent cone at a point

Let $X \subseteq k^n$ be a non-empty Zariski-closed subset.

Let $P \in k[T_1, \ldots, T_n]$ be a polynomial. For all $x \in k^n$, we have a Taylor expansion at x: For all $h \in k^n$:

$$\begin{aligned} P(x+h) &= P(x) + P'(x)h + \frac{1}{2}P''(x)(h,h) + \underbrace{\ldots}_{\text{finite number of terms}} \\ &= \sum_{d=0}^{\infty} \frac{1}{d!} P^{(d)}(x)(\underbrace{h, \ldots, h}_{d \text{ times}}). \end{aligned}$$

Remark 0.1. The term $\frac{1}{d!}P^{(d)}(x)$ is a homogeneous polynomial of degree d in the coordinates of $h = (h_1, \ldots, h_n)$:

$$P^{(d)}(x)(h, \ldots, h) = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| = d} \frac{d!}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{|\alpha|}}{\partial T_1^{\alpha_1} \cdots \partial T_n^{\alpha_n}} P(x) h_1^{\alpha_1} \cdots h_n^{\alpha_n}.$$

Also, when $x = 0_{k^n}$ and if we write

$$P = P(0) + \sum_{d=1}^{\infty} Q_d$$

with Q_d homogeneous of degree d, then for all $h = (h_1, \ldots, h_n) \in k^n$, we have

$$\frac{1}{d!} P^{(d)}(0) \cdot (h, \ldots, h) = Q_d(h_1, \ldots, h_n).$$

For all $P \in \mathcal{I}(X) \setminus \{0\}$, we denote by P_x^* the *initial term* in the Taylor expansion of P at x, i.e. the term $\frac{1}{d!}P^{(d)}(x) \cdot (h, \ldots, h)$ for the smallest $d \geq 1$ such that this is not zero. If $P = 0$, we put $P_x^* \coloneqq 0$.

Definition 0.2. We set

$$\mathcal{I}(X)_x^* \coloneqq \{P_x^* \colon P \in \mathcal{I}(X)\}.$$

Proposition 0.3. *The set $\mathcal{I}(X)_x^*$ is an ideal of $k[T_1, \ldots, T_n]$.*

Proof. By definition, $0 \in \mathcal{I}(X)_x^*$. Let P_x^*, Q_x^* be elements of $\mathcal{I}(X)_x^*$ coming from $P, Q \in \mathcal{I}(X)$. Then $P_x^* - Q_x^*$ is of the form R_x^* for some $R \in \mathcal{I}(X)$, where $R = 0$, $R = P$, $R = Q$ or $R = P - Q$. Moreover, for $Q \in k[T_1, \ldots, T_n]$, we have $P_x^* Q = (PQ)_x^* \in \mathcal{I}(X)_x^*$. □

Remark 0.4. The ideal $\mathcal{I}(X)^*$ is finitely generated. However, if $\mathcal{I}(X) = (P_1, \ldots, P_m)$, it is not true in general that $\mathcal{I}(X)_x^* = ((P_1)_x^*, \ldots, (P_m)_x^*)$. We may need to add the initial terms at x of some other polynomials of the form $\sum_{k=1}^m P_k Q_k \in \mathcal{I}(X)$.

If $\mathcal{I}(X) = (P)$ is principal though, we have $\mathcal{I}(X)_x^* = (P_x^*)$.

Definition 0.5. The *tangent cone* to X at x is the affine algebraic set

$$\mathcal{C}_x^{(X)} \coloneqq x + \mathcal{V}_{k^n}(\mathcal{I}(X)_x^*) = \{x + h \colon h \in \mathcal{V}_{k^n}(\mathcal{I}(X)_x^*)\}.$$

Remark 0.6. The algebraic set $\mathcal{C}_x(X)$ is a cone at x: It contains x and for all $x + h \in \mathcal{C}_x(X)$ for some $h \in \mathcal{V}_{k^n}(\mathcal{I}(X)_x^*)$, we have for all $\lambda \in k^\times$, $\lambda h \in \mathcal{V}_{k^n}(\mathcal{I}(X)_x^*)$, i.e. $x + \lambda h \in \mathcal{C}_x(X)$.

Indeed, $P_x^* \in \mathcal{I}(X)_x^*$ is either zero or a homogeneous polynomial of degree $r \geq 1$. Thus for $h \in k^n$ and $\lambda \in k^\times$: $P_x^*(\lambda h) = \lambda^r P_x^*(h)$ which is 0 if and only if $P_x^*(h) = 0$.

Example 0.7. Let k be an infinite field and let $P \in k[x,y]$ be an irreducible polynomial such that $X := \mathcal{V}(P)$ is infinite. Then we know that $\mathcal{I}(X) = (P)$. Then we can determine $\mathcal{C}_X(X)$ by computing the successive derviatives of P at x: In this case $\mathcal{I}(X)_x^* = (P_x^*)$. For convenience wie will mostly consider examples for which $x = 0_{k^2}$.

(i) $P(x,y) = y^2 - x^3$. Then $P_{(0,0)}^* = y^2$, so the tangent cone at $(0,0)$ is the algebraic set

$$\mathcal{C}_{(0,0)}(X) = \{(x,y) \in k^2 \mid y^2 = 0\}.$$

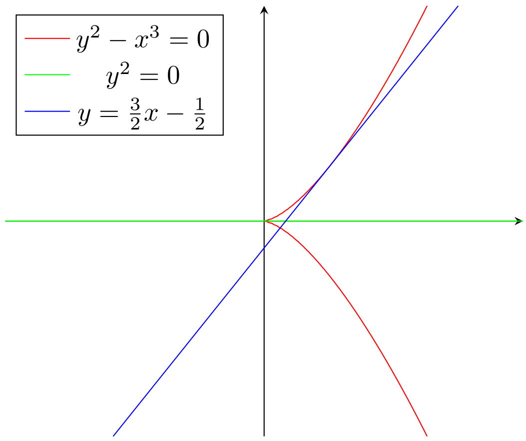

Figure 0.1: The green line is the tangent cone at $(0,0)$ and the blue line the tangent cone at $(1,1)$.

Note that $P_{(1,1)}^*(h_1,h_2) = 2h_2 - 3h_1$, so the tangent cone at $(1,1)$ is

$$\begin{aligned}\mathcal{C}_{(1,1)}(X) &= \{(1+h_1, 1+h_2) \mid 2h_2 - 3h_1 = 0\} \\ &= \left\{(x,y) \in k^2 \mid y = \frac{3}{2}x - \frac{1}{2}\right\}.\end{aligned}$$

(ii) $P(x,y) = y^2 - x^2(x+1)$. Then $P_{(0,0)}^* = y^2 - x^2$ so

$$\mathcal{C}_{(0,0)}(X) = \{y^2 - x^2 = 0\}$$

which is a union of two lines.

In contrast, $P_{(1,1)}^*(h_1,h_2) = 2h_2 - 5h_1$ so

$$\mathcal{C}_{(1,1)}(X) = \left\{(x,y) \in k^2 \mid y = \frac{5}{2}x - \frac{3}{2}\right\},$$

which is just one line. Evidently this is related to the origin being a „node“ of the curve of equation $y^2 - x^2(x+1) = 0$.

Remark 0.8. (i) The tangent cone $\mathcal{C}_x(X)$ represents all directions coming out of x along which the initial term P_x^* vanishes, for all $P \in \mathcal{I}(X)$. In that sense, it is the least complicated approximation to X around x, in terms of the degrees of the polynomials involved.

(ii) The notion of tangent cone at a point enables us to define singular points of algebraic sets and even distinguish between the type of singularities: Let $\mathcal{I}(X) = (P)$.

When $\deg(P_x^*) = 1$, the tangent cone to $X \subseteq k^n$ at x is just an affine hyperplane, namely $x + \ker P'(x)$, since $P_x^* = P'(x)$ in this case. The point x is then called *non-singular*.

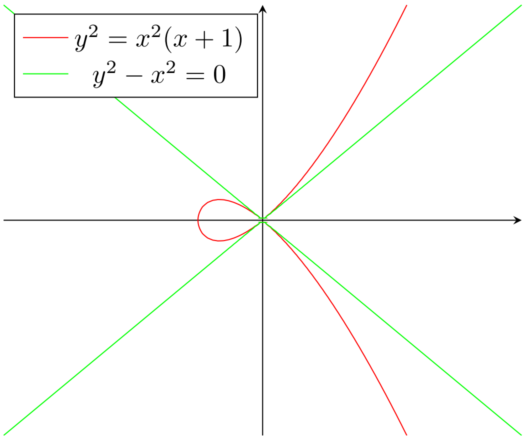

Figure 0.2: The green line is the tangent cone at $(0,0)$.

When $\deg(P_x^*) = 2$, we say that X has a *quadratic singularity* at x. If $X \subseteq k^2$, a quadratic singularity is called a *double point*. In that case, $P_x^* = \frac{1}{2}P''(x)$ is a quadratic form on k^2. If it is non-degenerate, then x is called an *ordinary* double point. For instance, if X is the nodal cubic of equation $y^2 = x^2(x+1)$, then the origin is an ordinary double point (also called a *node*), since $\frac{1}{2}P''(0,0)$ is the quadratic form associated to the symmetric matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. But if X is the cuspidal cubic of equation $y^2 = x^3$, then the origin is *not* an ordinary double point, since $\frac{1}{2}P''(0,0)$ corresponds to $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. Instead, the origin is a *cusp* in the following sense. We can write

$$P(x,y) = l(x,y)^2 + Q_3(x,y) + \dots$$

with $l(x,y) = \alpha x + \beta y$ a linear form in (x,y), and the double point $(0,0)$ is called a cusp if $Q_3(\beta, -\alpha) \neq 0$. This means that

$$t^4 X P(\beta t, -\alpha t)$$

in $k[t]$. And this is indeed what happens for $P(x,y) = y^2 - x^3$, since $l(x,y) = y$ and $Q_3(x,y) = -x^3$.

Remark 0.9. One can define the *multiplicity* of a point $(x,y) \in \mathcal{V}_{k^2}(P)$ as the smallest integer $r \geq 1$ such that $P^{(r)}(x,y) \neq 0$. If $P^{(r)}(x,y) \cdot (h, \dots, h) = 0 \implies h = 0_{k^2}$, the singularity (x,y) is called *ordinary*. If k is algebraically closed and $(x,y) = (0,0)$, we can write $P^{(r)}(0,0) = \prod_{i=1}^m (\alpha_i x + \beta_i y)^{r_i}$, with $r_1 + \dots + r_m = r$. Then $(0,0)$ is an ordinary singularity of multiplicity r iff $r_i = 1$ for all i. For instance, $(0,0)$ is an ordinary triple point of the trefoil curve $P(x,y) = (x^2+y^2)^2 + 3x^2y - y^3$.

0.1.2 The Zariski tangent space at a point

Let $X \subseteq k^n$ be a Zariski-closed subset and $x \in X$.

The tangent cone is in general not a linear approximation. To remedy this, one can consider the Zariski tangent space to X at a point $x \in X$.

Definition 0.10. The *Zariski tangent space* to X at x is the affine subspace

$$T_x X := x + \bigcap_{P \in \mathcal{I}(X)} \ker P'(x).$$

Remark 0.11. By translation, T_xX can be canonically identified to the vector space $\bigcap_{P\in\mathcal{I}(X)} \ker\, P'(x)$.

Proposition 0.12. *View the linear forms*

$$P'(x)\colon h \mapsto P'(x)\cdot h$$

as homogeneous polynomials of degree 1 *in the coordinates of* $h \in k^n$ *and denote by*

$$\mathcal{I}(X)_x \coloneqq (P'(x) : P \in \mathcal{I}(X))$$

the ideal generated by these polynomials. Then

$$T_xX = x + \mathcal{V}_{k^n}(\mathcal{I}(X)_x).$$

Proof. It suffices to check that

$$\mathcal{V}_{k^n}(\mathcal{I}(X)_x) = \bigcap_{P\in\mathcal{I}(X)} \ker\, P'(x)$$

which is obvious because the $(P'(x))_{P\in\mathcal{I}(X)}$ generate $\mathcal{I}(X)_x$. □

Corollary 0.13. $T_xX \supseteq \mathcal{C}_x(X)$

Proof. Since $\mathcal{I}(X)_x \subseteq \mathcal{I}(X)_x^*$, one has $\mathcal{V}_{k^n}(\mathcal{I}(X)_x) \supseteq \mathcal{V}_{k^n}(\mathcal{I}(X)_x^*)$. □

Definition 0.14. If $T_xX = \mathcal{C}_x(X)$, the point x is called *non-singular.*

Proposition 0.15. *If* $\mathcal{I}(X) = (P_1, \ldots, P_m)$*, then* $\mathcal{I}(X)_x = (P_1'(x), \ldots, P_m'(x))$

Proof. By definition,

$$(P_1'(x), \ldots, P_m'(x)) \subseteq (P'(x) : P \in \mathcal{I}(X)) = \mathcal{I}(X)_x.$$

But for $P \in \mathcal{I}(X)$, there exist $Q_1, \ldots, Q_m \in k[T_1, \ldots, T_n]$ such that $P = \sum_{i=1}^m Q_iP_i$, so

$$\begin{aligned} P'(x) &= \sum_{i=1}^m (Q_iP_i)'(x) \\ &= \sum_{i=1}^m (Q_i'(x)\underbrace{P_i(x)}_{=0} + \overbrace{Q_i(x)}^{\in k} P_i'(x)) \end{aligned}$$

since $x \in X$. This proves that $P'(x)$ is in fact a linear combination of the linear forms $(P_i'(x))_{1\le i\le m}$. □

Corollary 0.16. *If* $\mathcal{I}(X) = (P_1, \ldots, P_m)$*, then* $T_xX = x + \bigcap_{i=1}^m \ker P_i'(x)$*. Moreover, if we write* $P = (P_1, \ldots, P_m)$*, and view this* P *as a polynomial map* $k^n \to k^m$*, then*

$$T_xX = x + ker\ P'(x)$$

with $P'(x)$ *the Jacobian of* P *at* x*, i.e.*

$$P'(x) = \begin{pmatrix} \frac{\partial P_1}{\partial T_1}(x) & \cdots & \frac{\partial P_1}{\partial T_n}(x) \\ \vdots & & \vdots \\ \frac{\partial P_m}{\partial T_1}(x) & \cdots & \frac{\partial P_m}{\partial T_n}(x) \end{pmatrix}.$$

In particular, dim $T_xX = n - \operatorname{rk} P'(x)$.

Example 0.17. (i) $X = \{y^2 - x^3 = 0\} \subseteq k^2$. Then $\mathcal{I}(X) = (y^2 - x^3)$, so,

$$T_{(0,0)}X = (0,0) + \ker \begin{pmatrix} 0 & 0 \end{pmatrix} = k^2.$$

which strictly contains the tangent cone $\{y^2 = 0\}$. In particular, the origin is indeed a singular point of the cuspidal cubic. In general,

$$T_{(x,y)}X = (x,y) + \ker \begin{pmatrix} -3x^2 & 2y \end{pmatrix},$$

which is an affine line if $(x,y) \neq (0,0)$.

(ii) $X = \{y^2 - x^2 - x^3 = 0\} \subseteq k^2$. Then $\mathcal{I}(X) = (y^2 - x^2 - x^3)$, so

$$T_{(0,0)}X = (0,0) + \ker \begin{pmatrix} 0 & 0 \end{pmatrix} = k^2$$

which again strictly contains the tangent cone $\{y = \pm x\}$. In general,

$$T_{(x,y)}X = (x,y) + \ker \begin{pmatrix} -2x & 2y \end{pmatrix},$$

which is an affine line if $(x,y) \neq (0,0)$.

Remark 0.18. The dimension of the Zariski tangent space at x (as an affine subspace of k^n) may vary with x.

Proposition 0.19 (a Jacobian criterion)**.** *If $(P_1, \ldots, P_m)$ are polynomials such that $\mathcal{I}(X) = (P_1, \ldots, P_m)$ and $\operatorname{rk} P'(x) = m$, where $P = (P_1, \ldots, P_m)$, then x is a non-singular point of X.*

Proof. By 0.13 and 0.16 it suffices to show that

$$\mathcal{C}_x(X) \supseteq x + \bigcap_{i=1}^{m} \ker\, P_i'(x).$$

By definition

$$\mathcal{C}_x(X) = x + \mathcal{V}_{k^n}(\mathcal{I}(X)_x^*)$$

and $\mathcal{I}(X)_x^* = \{Q_x^* : Q \in \mathcal{I}(X)\}$. If $Q \in \mathcal{I}(X)$, there exist polynomials $Q_1, \ldots, Q_m$ such that $Q = \sum_{i=1}^m Q_i P_i$, so Q_x^* is a linear combination of the $(P_i)_x^*$. Since rk $(P_1'(x), \ldots, P_m'(x)) = m$, we have $P_i'(x) \neq 0$ for all i. So $(P_i)_x^* = P_i'(x)$ in the Taylor expansion of P_i at x. So Q_x^* is a linear combination of $(P_1'(x), \ldots, P_m'(x))$, which proves that if $h \in \bigcap_{i=1}^m \ker\, P_i'(x)$, then $Q_x^*(h) = 0$ for all $Q \in \mathcal{I}(X)$, hence $x + h \in \mathcal{C}_x(X)$. $\square$