

# Groupschemes

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# Chapter 1

## Introduction

Literature: Görtz-Wedhorn: Algebraic Geometry I and II

The goal of this lecture is a brief introduction to the theory of group schemes.

**Definition 1.1** (Group object). Let  $\mathcal{C}$  be a category with finite products. A *group object in  $\mathcal{C}$*  is the data  $(G, m, e, i)$  where

- $G$  is an object of  $\mathcal{C}$
- $m: G \times G \rightarrow G$  is the multiplication map
- $e: 1 \rightarrow G$  is the unit
- $i: G \rightarrow G$  is the inversion map

such that the following diagrams commute

$$\begin{array}{ccc}
 G \times G \times G & \xrightarrow{m \times \text{id}} & G \times G \\
 \downarrow \text{id} \times m & & \downarrow m \\
 G \times G & \xrightarrow{m} & G
 \end{array}
 , \quad
 \begin{array}{ccc}
 G \times G & \xrightarrow{m} & G \\
 \text{id} \times e \uparrow & \swarrow & \\
 G \times 1 & & 
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 G & \xrightarrow{\text{id} \times i} & G \times G \\
 \downarrow & & \downarrow m \\
 1 & \xrightarrow{e} & G
 \end{array}
 .$$

$G$  is called *commutative*, if additionally the diagram

$$\begin{array}{ccc}
 G \times G & \xrightarrow{\text{swap}} & G \times G \\
 \downarrow m & \swarrow m & \\
 G & & 
 \end{array}$$

commutes.

A *morphism of group objects*  $(G, m, e, i) \rightarrow (G', m', e', i')$  is a morphism  $f: G \rightarrow G'$  in  $\mathcal{C}$  such that the diagrams

$$\begin{array}{ccc}
 G \times G & \xrightarrow{f \times f} & G' \times G' \\
 \downarrow m & & \downarrow m' \\
 G & \xrightarrow{f} & G'
 \end{array}
 , \quad
 \begin{array}{ccc}
 G & \xrightarrow{f} & G' \\
 e \uparrow & \swarrow e' & \\
 1 & & 
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 G & \xrightarrow{f} & G' \\
 \downarrow i & & \downarrow i' \\
 G & \xrightarrow{f} & G'
 \end{array}
 .$$

This defines the category  $\text{Grp}(\mathcal{C})$  of group objects in  $\mathcal{C}$ .

**Example 1.2.** •  $\text{Grp}(\text{Set}) \simeq \text{Grp}$

- $\text{Grp}(\text{Grp}) \simeq \text{Ab}$
- $\text{Grp}(\text{Ab}) \simeq ?$

- $\text{Grp}(\text{Top}) \simeq$  topological Groups
- $\text{Grp}(\text{Mfd}_\infty) \simeq$  Lie Groups

**Definition 1.3** (group scheme). Let  $S$  be a scheme. An  $S$ -group scheme or an  $S$ -group is a group object in the category of schemes over  $S$ .

**Remark 1.4.** Let  $S$  be a scheme. The structure of a group scheme over  $S$  on a  $S$ -scheme  $G$  is equivalent to a factorisation of the functor of points

$$\begin{array}{ccc} \text{Sch}_S & \longrightarrow & \text{Set} \\ \downarrow & \nearrow & \\ \text{Grp} & & \end{array}$$

via the forgetful functor from groups to sets.

**Example 1.5.** Let  $S$  be a scheme.

- Let  $\Gamma$  be a group. Then  $G = \Gamma_S$  where  $G(T) := \{ \text{locally constant maps } T \rightarrow \Gamma \}$
- (additive group)  $\mathbb{G}_{a,S}$  where  $\mathbb{G}_{a,S}(T) = \mathcal{O}_T(T)$ . We have  $\mathbb{G}_{a,S} \simeq \mathbb{A}_S^1$ .
- (multiplicative group)  $\mathbb{G}_{m,S}$  where  $\mathbb{G}_{m,S}(T) := \mathcal{O}_T(T)^\times$ .
- (roots of unity)  $\mu_{n,S}$  ( $n \geq 1$ ) where  $\mu_{n,S}(T) = \{x \in \mathcal{O}_T(T)^\times \mid x^n = 1\}$ .
- $S = \text{Spec}(R)$ .  $\text{GL}_{n,R} = \text{Spec}(A)$  where  $A = R[T_{ij} \mid 1 \leq i, j \leq n][\det^{-1}]$  where  $\det = \sum_{\sigma \in S_n} \text{sgn}(\sigma) T_{1\sigma(1)} \cdots T_{n\sigma(n)}$ . We obtain  $\text{GL}_{n,S}$  by base changing  $\text{GL}_{n,\mathbb{Z}}$ .

**Lemma 1.6.** Let  $G$  be a  $S$ -group. Then  $G \rightarrow S$  is separated if and only if  $S \xrightarrow{e} G$  is a closed immersion.

**Definition 1.7.** Let  $R$  be a ring. A (commutative) Hopf-Algebra over  $R$  is a group object in  $\text{Alg}_R^{\text{op}}$ , where  $\text{Alg}_R = \text{CRing}_R$ .

**Remark 1.8.** For a  $R$ -Hopf-Algebra  $A$ , we denote the canonical maps by

- $\mu: A \rightarrow A \otimes_R A$  (Comultiplication)
- $\varepsilon: A \rightarrow R$  (Counit)
- $\iota: A \rightarrow A$  (Antipode)

A Hopf-Algebra is called *cocommutative*, if the associated group object in  $\text{Alg}_R^{\text{op}}$  kommutativ ist.

**Remark 1.9.** For a ring  $R$ , by construction we have an equivalence of categories between the category of affine  $R$ -group schemes and the opposite category of  $R$ -Hopf-Algebras.

**Example 1.10.** The additive group  $\mathbb{G}_{a,R} = \text{Spec}(R[t])$  has

- comultiplication  $\mu: R[t] \rightarrow R[t] \otimes_R R[t], t \mapsto t \otimes 1 - 1 \otimes t$ .
- counit  $\varepsilon: R[t] \rightarrow R, t \mapsto 0$
- antipode  $\iota: R[t] \rightarrow R[t], t \mapsto -t$

*Proof.* For any  $R$ -algebra  $A$  we have  $\mathbb{G}_{a,R}(A) = A$  and the diagram

$$\begin{array}{ccc} \mathbb{G}_{a,R}(A) \times \mathbb{G}_{a,R}(A) & \xrightarrow{m} & \mathbb{G}_{a,R}(A) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Hom}_R(R[s_1, s_2], A) & \xrightarrow{\mu^*} & \text{Hom}_R(R[t], A) \end{array} .$$

□

**Definition 1.11.** Let  $G$  be a  $S$ -group. A *subgroupscheme* of  $G$  is a subscheme  $H \subseteq G$  such that

- 1) for all  $T \in \text{Sch}_S$ , we have  $H(T) \subseteq G(T)$  a subgroup,
- 2) We have commutative diagrams

$$\begin{array}{ccc} H \times_S H & \longrightarrow & G \times_S G \xrightarrow{m} G \\ \downarrow & \nearrow & \\ H & & \end{array} \quad \text{and} \quad \begin{array}{ccc} S & \xrightarrow{e} & G \\ \downarrow & \nearrow & \\ H & & \end{array}$$

A subgroup scheme  $H \subseteq G$  is *normal* if  $H(T)$  is a normal subgroup of  $G(T)$  for all  $T \in \text{Sch}_S$ .

For a morphism  $f: G \rightarrow G'$  of  $S$ -groups and a subgroup  $H' \subseteq G'$ , let  $f^{-1}(H')$  be  $G \times'_G H'$ . For  $H' = 1 \xrightarrow{e} G'$ , we obtain the *kernel of  $f$*  and the cartesian square

$$\begin{array}{ccc} \text{Ker}(f) & \longrightarrow & G \\ \downarrow & & \downarrow f \\ S & \xrightarrow{e} & G' \end{array}$$

**Remark 1.12.** The kernel of a homomorphism  $f$  of  $S$ -groups is for any  $S$ -scheme  $T$  given by

$$\text{Ker}(f)(T) = \ker(f(T)).$$

In particular, the  $\text{Ker}(f)$  is normal.

**Definition 1.13.** Let  $G$  be a  $S$ -group,  $T$  a  $S$ -scheme and  $g \in G(T) = \text{Hom}_S(T, G)$ . Define the *lefttranslation by  $g$*  as

$$\begin{array}{ccc} G_T & \xleftarrow{=} & T \times_T G_T \\ \downarrow t_g & & \downarrow g \times \text{id} \\ G_T & \xleftarrow{m} & G_T \times_T G_T \end{array}$$

**Remark 1.14.** In the situation of 1.13, for every  $T' \xrightarrow{f} T$ , the map

$$t_g(T'): G_T(T') = G(T') \longrightarrow G(T') = G_T(T')$$

is the lefttranslation by the element  $f^*(g) \in G(T')$ .

**Remark 1.15.** Consider

$$\begin{array}{ccc} G \times_S G & \xrightarrow{(g,h) \mapsto (gh,h)} & G \times_S G \\ \downarrow m & \nearrow \text{pr}_1 & \\ G & & \end{array}$$

Let  $\mathcal{P}$  be a property of morphisms stable under base change and composition with isomorphisms. Then whenever  $G \rightarrow S$  satisfies  $\mathcal{P}$ , then  $m$  satisfies  $\mathcal{P}$ .