Chapter 1

Affine varieties

1.1 Spaces with functions

Definition 1.1. Let k be a field. A *space with functions over* k is a pair $(X, \mathcal{O}_x)$ where X is a topological space and $\mathcal{O}_X$ is a subsheaf of the sheaf of k-valued functions, seen as a sheef of k-algebras, and satisfying the following condition:

If $U \subseteq X$ is an open set and $f \in \mathcal{O}_X(U)$, then the set

$$D_U(f) \coloneqq \{x \in U \mid f(x) \neq 0\}$$

is open in U and the function $\frac{1}{f} \colon D_U(f) \to k$, $x \mapsto \frac{1}{f(x)}$ belongs to $\mathcal{O}_X(D_U(f))$.

Remark 1.2. Concretely, it means that there is for each open set $U \subseteq X$ a k-Algebra $\mathcal{O}_X(U)$ of „regular“ functions such that

(i) the restriction of a regular function $f \colon U \to k$ to a sub-open $U' \subseteq U$ is regular on U'.

(ii) if $f \colon U \to k$ is a function and $(U_\alpha)_{\alpha \in A}$ is an open cover of U such that $f|_{U_\alpha}$ is regular on U_α, then f is regular on U.

(iii) if f is regular on U, the set $\{f \neq 0\}$ is open in U and $\frac{1}{f}$ is regular wherever it is defined.

Remark 1.3. If $\{0\}$ is closed in k and $f \colon U \to k$ is continuous, then $D_U(f)$ is open in U. So, this conditions is often automatically met in practice.

Example 1.4. (i) $(X, \mathcal{C}_X)$ a topological space endowed with its sheaf of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) continuous functions, the fields $\mathbb{R}$ and $\mathbb{C}$ being endowed here with their classical topology.

(ii) $(V, \mathcal{O}_V)$ where $V = \mathcal{V}(P_1, \ldots, P_m)$ is an algebraic subset of k^n (endowed with the Zariski topology) and, for all $U \subseteq V$ open,

$$\mathcal{O}_V(U) \coloneqq \left\{ f \colon U \to k \;\middle|\; \begin{array}{l} \forall x \in U \exists x \in U_x \text{ open}, P, Q \in k[x_1, \ldots, x_n] \text{ such that} \\ \text{for } z \in U \cap U_x, Q(z) \neq 0 \text{ and } f(z) = \frac{P(z)}{Q(z)} \end{array} \right\}.$$

(iii) $(M, \mathcal{C}_M^\infty)$ where $M = \varphi^{-1}(0)$ is a non-singular level set of a $\mathcal{C}^\infty$ map $\varphi \colon \Omega \to \mathbb{R}^m$ where $\Omega \subseteq \mathbb{R}^{p+m}$ is an open set (in the usual topology of $\mathbb{R}^{p+m}$) and, for all $U \subseteq M$ open, $\mathcal{C}_M^\infty(U)$ locally smooth maps.

Exercise 1.5. Let $(X, \mathcal{O}_X)$ be a space with functions and let $U \subseteq X$ be an open subset. Define, for all $U' \subseteq U$ open,

$$\mathcal{O}_X|_U(U') \coloneqq \mathcal{O}_X(U').$$

Then $(U, \mathcal{O}_X|_U)$ is a space with functions.

Example 1.6. (i) $(V, \mathcal{O}_V)$ an algebraic subset of k^n, $f\colon V \to k$ a polynomial function, $U := D_V(f)$ is open in V and the sheaf of regular functions that we defined on the locally closed subset $D_V(f) = D_{k^n}(f) \cap V$ coincides with the restriction to $D_V(f)$ of the sheaf of regular functions on V.

(ii) $B \subseteq \mathbb{R}^n$ or $\mathbb{C}^n$ an open ball (with respect to the usual topology), equipped with the sheaf of $\mathcal{C}^\infty$ or holomorphic functions.

1.2 Morphisms

Remark 1.7. Note that if $f\colon X \to Y$ is a map and $h\colon U \to k$ is a function defined on a subset $U \subseteq Y$, there is a pullback map f_U^* taking $h\colon U \to k$ to the function $f_U^* := h \circ f\colon f^{-1}(U) \to k$. This map is a homomorphism of k-algebras. Moreover given a map $g\colon Y \to Z$ and a subset $V \subseteq Z$ such that $g^{-1}(V) \subseteq U$, we have, for all $h\colon V \to k$,

$$f_U^*(g_V^*(h)) = f_U^*(h \circ g) = (h \circ g) \circ f = h \circ (g \circ f) = (g \circ f)_V^*(h).$$

Definition 1.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two spaces with functions over a field k. A *morphism of spaces with functions* between $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ such that, for all open set $U \subseteq Y$, the pullback map f_U^* takes a regular function on the open set $U \subseteq Y$ to a regular function on the open set $f^{-1}(U) \subseteq X$.

Remark 1.9. Then, given open sets $U' \subseteq U$ in Y, we have compatible homomorphisms of k-algebras:

In other words, we have a morphism of sheaves on Y $f^*\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$, where by definition $(f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$.

Exercise 1.10. Given $g\colon Y \to Z$, show that $(g \circ f)_*\mathcal{O}_X = g_*(f_*\mathcal{O}_X)$ and that g_* is a functor from sheaves on Y to sheaves on Z.

Remark 1.11. If $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms, so is the composed map $g \circ f\colon X \to Z$.

Proposition 1.12. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally closed subsets of an affine space ($X \subseteq k^n, Y \subseteq K^m$) equipped with their respective sheaves of regular functions. Then a map $f\colon X \to Y$ is a morphism of spaces with functions if and only if $f = (f_1, \dots, f_m)$ with each $f_i\colon X \to k$ a regular function on X.*

Proof. The proof that if each of the f_i's is a regular function, then f is a morphism is similar to point (i) of the previous example: it holds because the pullback of a regular function (in particular, the pullback of a polynomial) by a regular function is a regular function, and because an equation of the form $h(x) = 0$ for h a regular function is locally equivalent to a polynomial equation $P(x) = 0$.

Conversely, if $f\colon X \to Y \subseteq k^m$ is a morphism, then the pullback of the i-th projection $p_i\colon k^m \to k$ is a regular function on X. Since $f^*p_i = f_i$, the proposition is proved. □

Remark 1.13. In the proof of the previous proposition, we used that if the $(f_i\colon X \to k)_{1 \le i \le m}$ are regular functions on the locally closed subset $X \subseteq k^n$, then the map

$$\begin{aligned} f\colon X &\to k^m \\ x &\mapsto (f_1(x), \dots, f_m(x)) \end{aligned}$$

is continuous on X. This is because the pre-image of $f^{-1}(V)$ of an algebraic subset $V = V(P_1, \dots, P_r) \subseteq k^m$ is the intersection of X with the zero set

$$W = V(P_1 \circ f, \dots, P_r \circ f) \subseteq k^n$$

which is indeed an algebraic set, because $P_j \circ f$ is a regular function so the equation $P_j \circ f = 0$ is equivalent to a polynomial equation.

Beware, however, that if the $(f_i)_{1 \leq i \leq m}$ are only continuous maps, then W is no longer an algebraic set, so we would need another argument in order to prove the continuity of f. Typically, in general topology, we say that $f\colon X \to k^m$ is continuous because its components $(f_1, \dots, f_m)$ are continuous. This argument is valid when the topology used on k^m is the product topology of the topologies on k. However, this does not hold in general for the Zariski topology, which is strictly larger than the product topology when k is infinite.

Example 1.14. (i) The projection map

$$\begin{aligned} \mathcal{V}_{k^2}(y - x^2) &\to k \\ (x, y) &\mapsto x \end{aligned}$$

is a morphism of spaces with functions, because it is a regular function on $\mathcal{V}_{k^2}(y - x^2)$. It is actually an isomorphism, whose inverse is the morphism

$$\begin{aligned} k &\to \mathcal{V}(y - x^2) \\ x &\mapsto (x, x^2). \end{aligned}$$

Note that $\mathcal{V}_{k^2}(y - x^2)$ is the graph of the polynomial function $x \mapsto x^2$.

(ii) Let k be an infinite field. The map

$$\begin{aligned} k &\to \mathcal{V}_{k^2}(y^2 - x^3) \\ t &\mapsto (t^2, t^3) \end{aligned}$$

is a morphism and a bijection, but it is not an isomorphism, because its inverse

$$\begin{aligned} \mathcal{V}_{k^2}(y^2 - x^3) &\to k \\ (x, y) &\mapsto \begin{cases} \frac{y}{x} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \end{aligned}$$

is not a regular map (this is where we use that k is infinite).

(iii) Consider the groups $G = \mathrm{GL}(n; k)$, $\mathrm{SL}(n; k)$, $\mathrm{O}(n; k)$, $\mathrm{SO}(n; k)$ etc. as locally closed subsets in k^{n^2} and equip them with their sheaves of regular functions. Then the multiplication $\mu\colon GxG \to G, (g_1, g_2) \mapsto g_1 g_2$ and and inversion $\iota\colon G \to G, g \mapsto g^{-1}$ are morphisms (here $G \times G$ is viewed as a locally closed subset of $k^{n^2} \times k^{n^2} \simeq k^{2n^2}$, equipped with its Zariski topology), since they are given by regular functions in the coefficients of the matrices.

Such groups will later be called *affine algebraic groups.*