Real algebraic varieties

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WiSe 2022

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Chapter 1

Affine varieties

1.1 Spaces with functions

Definition 1.1. Let k be a field. A *space with functions over* k is a pair $(X, \mathcal{O}_x)$ where X is a topological space and $\mathcal{O}_X$ is a subsheaf of the sheaf of k-valued functions, seen as a sheef of k-algebras, and satisfying the following condition:

If $U \subseteq X$ is an open set and $f \in \mathcal{O}_X(U)$, then the set

$$D_U(f) \coloneqq \{x \in U \mid f(x) \neq 0\}$$

is open in U and the function $\frac{1}{f} \colon D_U(f) \to k$, $x \mapsto \frac{1}{f(x)}$ belongs to $\mathcal{O}_X(D_U(f))$.

Remark 1.2. Concretely, it means that there is for each open set $U \subseteq X$ a k-Algebra $\mathcal{O}_X(U)$ of „regular" functions such that

(i) the restriction of a regular function $f \colon U \to k$ to a sub-open $U' \subseteq U$ is regular on U'.

(ii) if $f \colon U \to k$ is a function and $(U_\alpha)_{\alpha \in A}$ is an open cover of U such that $f|_{U_\alpha}$ is regular on U_α, then f is regular on U.

(iii) if f is regular on U, the set $\{f \neq 0\}$ is open in U and $\frac{1}{f}$ is regular wherever it is defined.

Remark 1.3. If $\{0\}$ is closed in k and $f \colon U \to k$ is continuous, then $D_U(f)$ is open in U. So, this conditions is often automatically met in practice.

Example 1.4. (i) $(X, \mathcal{C}_X)$ a topological space endowed with its sheaf of $\mathbb{R}$-valued (or $\mathbb{C}$-valued) continuous functions, the fields $\mathbb{R}$ and $\mathbb{C}$ being endowed here with their classical topology.

(ii) $(V, \mathcal{O}_V)$ where $V = \mathcal{V}(P_1, \ldots, P_m)$ is an algebraic subset of k^n (endowed with the Zariski topology) and, for all $U \subseteq V$ open,

$$\mathcal{O}_V(U) \coloneqq \left\{ f \colon U \to k \;\middle|\; \begin{array}{l} \forall x \in U \exists x \in U_x \text{ open}, P, Q \in k[x_1, \ldots, x_n] \text{ such that} \\ \text{for } z \in U \cap U_x, Q(z) \neq 0 \text{ and } f(z) = \frac{P(z)}{Q(z)} \end{array} \right\}.$$

(iii) $(M, \mathcal{C}^\infty_M)$ where $M = \varphi^{-1}(0)$ is a non-singular level set of a $\mathcal{C}^\infty$ map $\varphi \colon \Omega \to \mathbb{R}^m$ where $\Omega \subseteq \mathbb{R}^{p+m}$ is an open set (in the usual topology of $\mathbb{R}^{p+m}$) and, for all $U \subseteq M$ open, $\mathcal{C}^\infty_M(U)$ locally smooth maps.

Exercise 1.5. Let $(X, \mathcal{O}_X)$ be a space with functions and let $U \subseteq X$ be an open subset. Define, for all $U' \subseteq U$ open,

$$\mathcal{O}_X|_U(U') \coloneqq \mathcal{O}_X(U').$$

Then $(U, \mathcal{O}_X|_U)$ is a space with functions.

Example 1.6. (i) $(V, \mathcal{O}_V)$ an algebraic subset of k^n, $f\colon V \to k$ a polynomial function, $U := D_V(f)$ is open in V and the sheaf of regular functions that we defined on the locally closed subset $D_V(f) = D_{k^n}(f) \cap V$ coincides with the restriction to $D_V(f)$ of the sheaf of regular functions on V.

(ii) $B \subseteq \mathbb{R}^n$ or $\mathbb{C}^n$ an open ball (with respect to the usual topology), equipped with the sheaf of $\mathcal{C}^\infty$ or holomorphic functions.

1.2 Morphisms

Remark 1.7. Note that if $f\colon X \to Y$ is a map and $h\colon U \to k$ is a function defined on a subset $U \subseteq Y$, there is a pullback map f_U^* taking $h\colon U \to k$ to the function $f_U^* := h \circ f\colon f^{-1}(U) \to k$. This map is a homomorphism of k-algebras. Moreover given a map $g\colon Y \to Z$ and a subset $V \subseteq Z$ such that $g^{-1}(V) \subseteq U$, we have, for all $h\colon V \to k$,

$$f_U^*(g_V^*(h)) = f_U^*(h \circ g) = (h \circ g) \circ f = h \circ (g \circ f) = (g \circ f)_V^*(h).$$

Definition 1.8. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two spaces with functions over a field k. A *morphism of spaces with functions* between $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ such that, for all open set $U \subseteq Y$, the pullback map f_U^* takes a regular function on the open set $U \subseteq Y$ to a regular function on the open set $f^{-1}(U) \subseteq X$.

Remark 1.9. Then, given open sets $U' \subseteq U$ in Y, we have compatible homomorphisms of k-algebras:

In other words, we have a morphism of sheaves on Y $f^*\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$, where by definition $(f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$.

Exercise 1.10. Given $g\colon Y \to Z$, show that $(g \circ f)_*\mathcal{O}_X = g_*(f_*\mathcal{O}_X)$ and that g_* is a functor from sheaves on Y to sheaves on Z.

Remark 1.11. If $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms, so is the composed map $g \circ f\colon X \to Z$.

Proposition 1.12. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally closed subsets of an affine space ($X \subseteq k^n, Y \subseteq K^m$) equipped with their respective sheaves of regular functions. Then a map $f\colon X \to Y$ is a morphism of spaces with functions if and only if $f = (f_1, \dots, f_m)$ with each $f_i\colon X \to k$ a regular function on X.*

Proof. The proof that if each of the f_i's is a regular function, then f is a morphism is similar to point (i) of the previous example: it holds because the pullback of a regular function (in particular, the pullback of a polynomial) by a regular function is a regular function, and because an equation of the form $h(x) = 0$ for h a regular function is locally equivalent to a polynomial equation $P(x) = 0$.

Conversely, if $f\colon X \to Y \subseteq k^m$ is a morphism, then the pullback of the i-th projection $p_i\colon k^m \to k$ is a regular function on X. Since $f^*p_i = f_i$, the proposition is proved. □

Remark 1.13. In the proof of the previous proposition, we used that if the $(f_i\colon X \to k)_{1 \leq i \leq m}$ are regular functions on the locally closed subset $X \subseteq k^n$, then the map

$$\begin{aligned} f\colon X &\to k^m \\ x &\mapsto (f_1(x), \dots, f_m(x)) \end{aligned}$$

is continuous on X. This is because the pre-image of $f^{-1}(V)$ of an algebraic subset $V = V(P_1, \dots, P_r) \subseteq k^m$ is the intersection of X with the zero set

$$W = V(P_1 \circ f, \dots, P_r \circ f) \subseteq k^n$$

which is indeed an algebraic set, because $P_j \circ f$ is a regular function so the equation $P_j \circ f = 0$ is equivalent to a polynomial equation.

Beware, however, that if the $(f_i)_{1\leq i\leq m}$ are only continuous maps, then W is no longer an algebraic set, so we would need another argument in order to prove the continuity of f. Typically, in general topology, we say that $f\colon X \to k^m$ is continuous because its components $(f_1, \dots, f_m)$ are continuous. This argument is valid when the topology used on k^m is the product topology of the topologies on k. However, this does not hold in general for the Zariski topology, which is strictly larger than the product topology when k is infinite.

Example 1.14. (i) The projection map

$$\begin{aligned} \mathcal{V}_{k^2}(y - x^2) &\to k \\ (x, y) &\mapsto x \end{aligned}$$

is a morphism of spaces with functions, because it is a regular function on $\mathcal{V}_{k^2}(y - x^2)$. It is actually an isomorphism, whose inverse is the morphism

$$\begin{aligned} k &\to \mathcal{V}(y - x^2) \\ x &\mapsto (x, x^2). \end{aligned}$$

Note that $\mathcal{V}_{k^2}(y - x^2)$ is the graph of the polynomial function $x \mapsto x^2$.

(ii) Let k be an infinite field. The map

$$\begin{aligned} k &\to \mathcal{V}_{k^2}(y^2 - x^3) \\ t &\mapsto (t^2, t^3) \end{aligned}$$

is a morphism and a bijection, but it is not an isomorphism, because its inverse

$$\begin{aligned} \mathcal{V}_{k^2}(y^2 - x^3) &\to k \\ (x, y) &\mapsto \begin{cases} \frac{y}{x} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \end{aligned}$$

is not a regular map (this is where we use that k is infinite).

(iii) Consider the groups $G = \mathrm{GL}(n; k)$, $\mathrm{SL}(n; k)$, $\mathrm{O}(n; k)$, $\mathrm{SO}(n; k)$ etc. as locally closed subsets in k^{n^2} and equip them with their sheaves of regular functions. Then the multiplication $\mu\colon GxG \to G, (g_1, g_2) \mapsto g_1 g_2$ and and inversion $\iota\colon G \to G, g \mapsto g^{-1}$ are morphisms (here $G \times G$ is viewed as a locally closed subset of $k^{n^2} \times k^{n^2} \simeq k^{2n^2}$, equipped with its Zariski topology), since they are given by regular functions in the coefficients of the matrices.

Such groups will later be called *affine algebraic groups.*

1.3 Abstract affine varieties

Recall that an isomorphism of spaces with functions is a morphism $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ that admits an inverse morphism.

Remark 1.15. As we have seen, a bijective morphism is not necessarily an isomorphism.

Remark 1.16. Somewhat more formally, one could also define a morphism of spaces with functions (over k) to be a pair (f, φ) such that $f\colon X \to Y$ is a continuous map and $\varphi\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is the morphism of sheaves f^*. The question then arises how to define properly the composition $(g, \psi) \circ (f, \varphi)$. The formal answer is $(g \circ f, f_*(\varphi) \circ \psi)$.

Definition 1.17. Let k be a field. An (abstract) *affine variety over k* (also called an affine k-variety) is a space with functions $(X, \mathcal{O}_X)$ over k that is isomorphic to the space with functions $(V, \mathcal{O}_V)$, where V is an algebraic subset of some affine space k^n and $\mathcal{O}_V$ is the sheaf of regular functions on V.

A morphism of affine k-varieties is a morphism of the underlying spaces with functions.

Example 1.18. (i) An algebraic subset $V \subseteq k^n$, endowed with its sheaf of regular functions $\mathcal{O}_V$, is an affine variety.

(ii) It is perhaps not obvious at first, but a standard open set $D_V(f)$, where $f\colon V \to k$ is a regular function on an algebraic set $V \subseteq k^n$, defines an affine variety. Indeed, when equipped with its sheaf of regular functions, $D_V(f) \simeq \mathcal{V}_{k^{n+1}}(tf(x) - 1)$.

Remark 1.19. Let $(X, \mathcal{O}_X)$ be a space with functions. An open subset $U \subseteq X$ defines a space with functions $(U, \mathcal{O}_U)$. If $(U, \mathcal{O}_U)$ is isomorphic to some standard open set $D_V(f)$ of an algebraic set $V \subseteq k^n$, we will call U an *affine open set*.

Then the observation is the following: since an algebraic set $V \subseteq k^n$ is a finite union of standard open sets, every point x in an affine variety X has an affine open neighbourhood.

Less formally, an affine variety X, locally „looks like“ a standard open set $D_V(f) \subseteq k^n$, where $V \subseteq k^n$ is an algebraic set. In particular, open subsets of an affine variety also locally look like standard open sets. In fact, they are finite unions of such sets.

Example 1.20. The algebraic group $\mathrm{GL}(n;k)$ is an affine variety over k.

Remark 1.21. An algebraic set $(V, \mathcal{O}_V)$ is a subset $V \subseteq k^n$ defined by polynomial equations and equipped with its sheaf of regular functions. An affine variety $(X, \mathcal{O}_X)$ is „like an algebraic set“ but without a reference to a particular „embedding“ in affine space. This is similar to having a finitely generated k-Algebra A without specifying a particular isomorphism

$$A \simeq k[X_1, \dots, X_n]/I.$$

The next example will illustrate precisely this fact.

Example 1.22. Let us now give an abstract example of an affine variety. We consider a finitely generated k-algebra A and define $X := \mathrm{Hom}_{k-\mathrm{Alg}}(A, k)$. The idea is to think of X as points on which we can evaluate elements of A, which are thought of as functions on X. For $x \in \mathrm{Hom}_k(A, k)$ and $f \in A$ we set $f(x) := x(f) \in k$.

- Topology on X: for all ideal $I \subseteq A$, set

  $$\mathcal{V}_X(I) := \{x \in X \mid \forall x \in I\colon f(x) = 0\}.$$

  These subsets of X are the closed sets of a topology on X, which we may call the Zariski topology.

- Regular functions on X: if $U \subseteq X$ is open, a function $h\colon U \to k$ is called regular at $x \in U$ if there it exists an open set $x \in U_x$ and elements $P, Q \in A$ such that for $y \in U_x$, $Q(y) \neq 0$ and $h(y) = \frac{P(y)}{Q(y)}$ in k.

  The function h is called regular on U iff it is regular at $x \in U$. Regular functions then form a sheaf of k-algebras on X.

  Moreover, if $h\colon U \to k$ is regular on X, the set $D_X(h) := \{x \in X \mid h(x) \neq 0\}$ is open in X and the function $\frac{1}{h}$ is regular on $D_X(h)$.

So, we have defined a space with functions $(X, \mathcal{O}_X)$, at least whenever $X \neq \emptyset$. We show that X is an affine variety.

Proof. Fix a system of generators of A, i.e.

$$A \simeq k[t_1, \dots, t_n]/I$$

where $k[t_1, \dots, t_n]$ is a polynomial algebra. We denote by $\overline{t_1}, \dots, \overline{t_n}$ the images of $t_1, \dots, t_n$ in A and we define

$$\begin{aligned} \varphi\colon X = \mathrm{Hom}_k(A, k) &\to k^n \\ x &\mapsto (x(\overline{t_1}), \dots, x(\overline{t_n})). \end{aligned}$$

Let $P \in I$ and $x \in X$. Then

$$P(\varphi(x)) = P(x(\overline{t_1}), \dots, x(\overline{t_n})) = x(\overline{P}) = 0.$$

Thus $\varphi(x) \in \mathcal{V}_{k^n}(I)$. Conversely let $a = (a_1, \dots, a_n) \in \mathcal{V}_{k^n}(I)$, then we can define a morphism of k-algebras

$$x_a\colon A \to A/(\overline{t_1} - a_1, \dots, \overline{t_n} - a_n) \simeq k$$

which satisfies $x_a(\overline{t_i}) = a_i$ for all i. So $(a_1, \dots, a_n) = \varphi(x_a) \in \mathrm{im}\ \varphi$.

In particular, we have defined a map

$$\begin{aligned} \psi\colon \mathcal{V}_{k^n}(I) &\to X = \mathrm{Hom}_k(A, k) \\ a &\mapsto x_a \end{aligned}$$

such that $\varphi \circ \psi = \mathrm{Id}_{\mathcal{V}_{k^n}(I)}$. In fact, we also have $\psi \circ \varphi = \mathrm{Id}_X$.

It remains to check that φ and ψ are morphisms of spaces with functions, which follows from the definition of the topology and the notion of regular function on X. □

The elements of $X := \mathrm{Hom}_k(A, k)$ are also called the *characters* of the k-algebra A, and this is sometimes denoted by $\hat{A} := \mathrm{Hom}_{k-\mathrm{alg}}(A, k)$. Note that $\hat{A}$ is a k-subalgebra of the algebra of all functions $f\colon A \to k$.

The character x_a introduced above and associated to an alemenet $a \in A$ is then denoted by $\hat{a}$ and called the *Gelfand transform* of a. The *Gelfand transformation* is the morphism of k-algebras

$$\begin{aligned} A &\to \hat{A} \\ a &\mapsto \hat{a}. \end{aligned}$$

Exercise 1.23. Let A be a finitely generated k-algebra and let $X = \mathrm{Hom}_{k\text{-alg}}(A, k)$. Show that the map $x \mapsto \ker x$ induces a bijection

$$X \simeq \{\mathfrak{m} \in \mathrm{Spm}\, A \mid A/\mathfrak{m} \simeq k\}.$$

Remark 1.24. Note that we have not assumed A to be reduced and that, if we set $A_{\mathrm{red}} := A/\sqrt{(0)}$, then A_{red} is reduced and $\hat{A_{\mathrm{red}}} = \hat{A}$, because a maximal ideal of A necessarily contains $\sqrt{(0)}$ and the quotient field is „the same“.

Remark 1.25. Let $(X, \mathcal{O}_X)$ be an affine variety. One can associate the k-algebra $\mathcal{O}_X(X)$ of globally defined regular functions on X:

$$\mathcal{O}_X(X) = \{f\colon X \to k \mid f \text{ regular on } X\}.$$

Moreover, if $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism between two affine varieties, we have a k-algebra homomorphism

$$\begin{aligned} \varphi^*\colon \mathcal{O}_Y(Y) &\to \mathcal{O}_X(X) \\ f &\mapsto f \circ \varphi. \end{aligned}$$

Also, $(\mathrm{id}_X)^* = \mathrm{id}_{\mathcal{O}_X(X)}$ and $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ whenever $\psi\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ is a morphism of affine varieties. In other words, we have defined a (contravariant) functor k-Aff $\to$ k-Alg.

Proposition 1.26. *Let k be a field. The functor*

$$k\text{-}\mathit{Aff} \to k\text{-}\mathit{Alg}$$
$$(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$$

is fully faithful.

Proof. Since X and Y are affine, we may assume $X = V \subseteq k^n$ and $Y = W \subseteq k^m$. Then $\varphi\colon V \to W$ is given by m regular functions $(\varphi_1, \ldots, \varphi_m)$ on V. On k^m, let us denote by y_i the projection to the i-th factor. Its restriction to W is a regular function

$$y_i|_W : W \to k$$

that satisfies $\varphi^*(y_i|_W) = \varphi_i$.

Since for all regular functions $f\colon W \to k$ one has

$$\varphi^* f = f \circ \varphi = f(\varphi_1, \ldots, \varphi_m),$$

we see that the morphism

$$\varphi^* : \mathcal{O}_W(W) \to \mathcal{O}_V(V)$$

is entirely determined by the m regular functions $\varphi^*(y_i|_W) = \varphi_i$ on V. In particular, if $\varphi^* = \psi^*$, then $\varphi_i = \varphi^*(y_i|_W) = \psi^*(y_i|_W) = \psi_i$, so $\varphi = \psi$, which proves that $\varphi \mapsto \varphi^*$ is injective.

Surjectivity: Let $h\colon \mathcal{O}_W(W) \to \mathcal{O}_V(V)$ be a morphism of k-algebras. Let

$$\varphi := (h(y_1|_W), \ldots, h(y_m|_W))$$

which is a morphism from V to k^m, because its components are regular functions on V. It satisfies $\varphi^*(y_i|_W) = \varphi_i = h(y_i|_W)$, so $\varphi^* = h$.

It remains to show, that $\varphi(V) \subseteq W$. Let $W = \mathcal{V}(P_1, \ldots, P_r)$ with $P_j \in k[Y_1, \ldots, Y_m]$. Then for all $j \in \{1, \ldots, r\}$ and $x \in V$

$$P_j(\varphi(x)) = P_j(h(y_1|_W), \ldots, h(y_m|_W))(x).$$

Since h is a morphism of k-algebras and P_j is a polynomial, we have

$$P_j(h(y_1|_W), \ldots, h(y_m|_W)) = h(P_j(y_1|_W), \ldots, P_j(y_m|_W)).$$

But $P_j \in \mathcal{I}(W)$, so

$$P_j(y_1|_W, \ldots, y_m|_W) = P_j(y_1, \ldots, y_m)|_W = 0,$$

which proves that for $x \in V$, $\varphi(x) \in W$. □

1.4 Geometric Noether normalisation

Consider a plane algebraic curve $\mathcal{C}$, defined by the equation $f(x, y) = 0$. If we fix $x = a$, then the polynomial equation $f(a, y) = 0$ has only finitely many solutions (at most $\deg_y f$). This means that the map

$$\mathcal{C} := \mathcal{V}(f) \to k(x, y) \mapsto x$$

has finite fibres. A priori, such a map is not surjective, e.g. for $f(x, y) = xy - 1$. If k is algebraically closed, one can always find such a surjective projection.

Theorem 1.27. *Let k be an algebraically closed field and $f \in k[x_1, \ldots, x_n]$ be a polynomial of degree $d \geq 1$. Then there is a morphism of affine varieties*

$$\pi\colon \mathcal{V}_{k^n}(f) \to k^{n-1}$$

such that:

(i) π *is surjective*

(ii) for $t \in k^{n-1}$*, the fibre* $\pi^{-1}(\{t\}) \subseteq \mathcal{V}(f)$ *consists of at most* d *points.*

Proof. Let $f \in k[x_1, \dots, x_n]$ be of degree d. We construct a change of variables of the form $(x_i \mapsto x_i + a_i x_n)_{1 \le i \le n-1}$ and $x_n \mapsto x_n$, such that the term of degree d of $f(x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1}x_n, x_n)$ becomes cx_n^d with $c \in k^\times$. Since

$$f(x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1}x_n, x_n) = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} \alpha_{i_1,\dots,i_n}(x_1 + a_1 x_n)^{i_1} \cdots (x_{n-1} + a_{n-1}x_n)^{i_{n-1}} x_n^{i_n},$$

the coefficient of x_n^d in the above equation is obtained by considering all $(i_1, \dots, i_n)$ such that $i_1 + \ldots + i_n = d$, and keeping only the term in $x_n^{i_j}$ when expanding $(x_j + a_j x_n)^{i_j}$, so we get

$$\sum_{(i_1,\dots,i_n)\in\mathbb{N} i_1+\ldots+i_n=d} \alpha_{i_1,\dots,i_n} a_1^{i_1} \cdots a_{n-1}^{i_{n-1}},$$

which is equal to $f_d(a_1, \dots, a_{n-1}, 1)$, where f_d is the (homogeneous) degree d part of f.

Claim: There exist $a_1, \dots, a_{n-1} \in k$ such that $f_d(a_1, \dots, a_{n-1}, 1) \neq 0$. Proof of claim by induction: if $n = 1$, $f_d = cx_1^d$ for some $c \neq 0$, so $f_d(1) = c \neq 0$. If $n \geq 2$, we can write

$$f_d(x_1, \dots, x_n) = \sum_{i=0}^{d} h_i(x_2, \dots, x_n) x_1^i$$

where $h_i \in k[x_2, \dots, x_n]$ is homogeneous of degree $d - i$. Since $f_d \neq 0$, there is at least one i_0 such that $h_{i_0} \neq 0$. By induction, we can find $(a_2, \dots, a_{n-1}) \in k^{n-2}$ such that $h_{i_0}(a_2, \dots, a_{n-1}, 1) \neq 0$. But then $f(\cdot, a_2, \dots, a_{n-1}, 1) \in k[x_1]$ is a non zero polynomial, so it has only finitely many roots. As k is infinite, there exists $a_1 \in k$, such that $f(a_1, \dots, a_{n-1}, 1) \neq 0$.

Then

$$\varphi\colon \begin{cases} x_i \mapsto x_i + a_i x_n & 1 \le i \le n-1 \\ x_n \mapsto x_n \end{cases}$$

is a invertible linear transformation $k^n \to k^n$, such that

$$(f \circ \varphi^{-1})(y_1, \dots, y_n) = c(y_n^d + g_1(y_1, \dots, y_n) y_n^{d-1} + \ldots + g_d(y_1, \dots, y_{n-1})$$

for $c \neq 0$. This induces an isomorphism of affine varieties

$$\begin{aligned} \mathcal{V}(f) &\to \mathcal{V}(f \circ \varphi^{-1}) \\ x &\mapsto \varphi(x) \end{aligned}$$

such that

$$\begin{array}{ccc} \mathcal{V}(f) & \overset{\varphi}{\hookrightarrow} & k^n = k^{n-1} \times k \\ & \overset{\pi}{\searrow} & \downarrow \\ & & k^{n-1} \end{array}$$

defines the morphism π with the desired properties. Indeed: Let $(x_1, \dots, x_n) \in k^n$ and set $y_i := \varphi(x_i)$. Then

$(x_1, \dots, x_n) \in \mathcal{V}(f)$ iff $x_n = y_n$ is a root of the polynomial

$$t^d + \sum_{j=1}^{d} g_j(y_1, \dots, y_{n-1}) t^{d-j}.$$

Therefore for all $t = (y_1, \dots, y_{n-1}) \in k^{n-1}$, $\pi^{-1}(\{t\}) \neq \emptyset$ (because $\overline{k} = k$) and $\pi^{-1}(\{t\})$ has at most d points. $\square$

Definition 1.28. Let $f \in k[x_1, \dots, x_n]$ be a polynomial of degree d. As in the proof of 1.27, ther exists a linear coordinate transformation $\varphi\colon k^n \to k^n$, such that $f \circ \varphi^{-1}(y_1, \dots, y_n) = cy_n^d + \sum_{j=1}^d g_j(y_1, \dots, y_{n-1})y_n^{d-j}$. For a point $x \in \pi^{-1}(y_1, \dots, y_{n-1}) \subseteq \mathcal{V}(f)$, the *multiplicity* of x is the multiplicity of y_n as a root of that polynomial.

A point with multiplicity ≥ 2 are called *ramification point* and its image lies in the *discriminant locus* of π.

With this vocabulary, we can refine the statement of 1.27.

Definition 1.29 (Geometric Noether normalisation)**.** Assume $k = \overline{k}$. If $f \in k[x_1, \dots, x_n]$ is polynomial of degree d, a morphism of affine varieties

$$\pi\colon \mathcal{V}_{k^n}(f) \to k^{n-1}$$

such that

(i) π is surjective

(ii) for $t \in k^{n-1}$, the number of elements in $\pi^{-1}(\{t\})$, counted with their respective multiplicities, is exactly d,

is called a *geometric Noether normalisation.*

Corollary 1.30 (Geometric Noether normalisation for hypersurfaces)**.** *Let k be an algebraically closed field and $f \in k[x_1, \dots, x_n]$ be a polynomial of degree $d \geq 1$. Then there exists a geometric Noether normalisation.*

Example 1.31. Let $f(x, y) = y^2 - x^3 \in \mathbb{C}[x, y]$. Then the map

$$\mathcal{V}_{\mathbb{C}^2}(y^2 - x^3) \to \mathbb{C}(x, y) \qquad\qquad \mapsto y$$

is a geometric Noether normalisation, but $(x, y) \mapsto x$ is not (the fibres of the latter have degree 2, while $\deg f = 3$).

Remark 1.32. In the proof of 1.27, to construct φ and the g_j, we only used that k is infinte. Thus the statement, that for all $f \in k[x_1, \dots, x_n]$ there exists a linear automorphism $\varphi\colon k^n \to k^n$ such that

$$f \circ \varphi^{-1}(y_1, \dots, y_n) = c\left(y_n^d + \sum_{j=1}^d g_j(y_1, \dots, y_{n-1})y_n^{d-j}\right)$$

is valid over k if k is infinite. The resulting map

$$\pi\colon \mathcal{V}_{k^n}(f) \to k^{n-1}$$

still has finite fibres, but it is no longer surjective in general, as the example $f(x, y) = x^2 + y^2 - 1$ shows.

However, it induces a surjective map with finite fibres

$$\hat{\pi}\colon \mathcal{V}_{\overline{k}^n}(f) \to \overline{k}^{n-1}$$

which moreover commutes with the action of $\mathrm{Gal}(\overline{k}/k)$.

Theorem 1.33. *Let k be an infinite field and $\overline{k}$ an algebraic closure of k. Let $f \in k[x_1, \dots, x_n]$ be a polynomial of degree $d \geq 1$. Then there exists a Gal($\overline{k}/k$)-equivariant geometric Noether normalisation map $\pi\colon \mathcal{V}_{\overline{k}^n}(f) \to \overline{k}^{n-1}$.*

Example 1.34. Let $f(x, y) = y^2 - x^3 \in \mathbb{R}[x, y]$. Then the map

$$\begin{aligned} \pi \colon \mathcal{V}_{\mathbb{C}^2}(y^2 - x^3) &\to \mathbb{C} \\ (x, y) &\mapsto y. \end{aligned}$$

is a geometric Noether normalisation map and it is Galois-invariant:

$$\pi(\overline{(x, y)}) = \pi(\overline{x}, \overline{y}) = \overline{y} = \overline{\pi(x, y)}.$$

Exercise 1.35. Show that if $y \in \mathbb{R}$, the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\pi^{-1}(\{y\})$, and that the fixed point set of that action is in bijection with $\{x \in \mathbb{R} \mid y^2 - x^3 = 0\}$.

Next, we want to generalise the results above beyond the case of hypersurfaces.

Theorem 1.36. *Assume k is algebraically closed. Let $V \subseteq k^n$ be an algebraic set. Then there exists a natural number $r \leq n$ and a morphism of algebraic sets*

$$p \colon V \to k^r$$

such that p is surjective and has finite fibres.

Sketch of proof. If $V = k^n$, we take $r = n$ and $p = \mathrm{id}_{k^n}$. Otherwise $V = \mathcal{V}(I)$ with $I \subseteq k[x_1, \ldots, x_n]$ a non-zero ideal. Take $f \in I \setminus \{0\}$. Then there exists a geometric Noether normalisation

$$p_1 \colon \mathcal{V}(f) \to k^{n-1}.$$

One can now show that $V_1 := p_1(V)$ is an algebraic set in k^{n-1}. Thus there are two cases:

(1) $p_1(V) = k^{n-1}$. Thus $p_1|_V \colon V \to k^{n-1}$ is surjective with finite fibres and we are done.

(2) $p_1(V) \subsetneq k^{n-1}$. In this case $p_1(V) = \mathcal{V}(I_1)$ with $I_1 \subseteq k[x_1, \ldots, x_{n-1}]$ a non-zero ideal. So we can repeat the argument.

After $r \leq n$ steps, the above algorithm terminates, and this happens precisely when $V_r = k^{n-r}$. If we set

$$p := p_r \circ \ldots \circ p_1 \colon V \to k^{n-r}$$

then p is surjective with finite fibres because $p(V) = V_r = k^{n-r}$ and each p_i has finite fibres. □

Remark 1.37. By the fact used in the proof of 1.36, p is in fact a closed map. Note that when $r = n$, $V = p^{-1}(\{0\})$ is actually finite, in which case dim V should indeed be 0.

1.5 Gluing spaces with functions

We present a general technique to construct spaces with functions by „patching together“ other spaces with functions „along open subsets“. This will later be used to argue that, in order to define a structure of variety on a topological sapce (or even a set), it suffices to give one atlas.

Theorem 1.38 (Gluing theorem)**.** *Let $(X_i, \mathcal{O}_{X_i})_{i \in I}$ be a family of spaces with functions. For all pair (i, j), assume that the following has been given*

(a) an open subset $X_{ij} \subseteq X_i$

(b) an isomorphism of spaces with functions

$$\varphi_{ji} \colon (X_{ij}, \mathcal{O}_{X_{ij}}) \to (X_{ji}, \mathcal{O}_{X_{ji}})$$

subject to the following compatibility conditions

(1) for all i, $X_{ii} = X_i$ and $\varphi_{ii} = id_{X_i}$

(2) for all pair (i,j), $\varphi_{ij} = \varphi_{ji}^{-1}$

(3) for all triple (i,j,k), $\varphi_{ji}(X_{ik} \cap X_{ij}) = X_{jk} \cap X_{ji}$ and $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ on $X_{ik} \cap X_{ij}$.

Then there exists a space with functions $(X, \mathcal{O}_X)$ equipped with a family of open sets $(U_i)_{i\in I}$ and isomorphisms of spaces with functions

(A1) $\varphi_i \colon (U_i, \mathcal{O}_X|_{U_i}) \to (X_i, \mathcal{O}_{X_i})$,

such that $\bigcup_{i\in I} U_i = X$ and, for all pair (i,j),

(B2) $\varphi_i(U_i \cap U_j) = X_{ij}$, and

(C3) $\varphi_j \circ \varphi_i^{-1} = \varphi_{ji}$ on X_{ij}.

Such a familiy $(U_i, \varphi_i)_{i\in I}$ is called an atlas for $(X, \mathcal{O}_X)$.

Moreover, if $(Y, \mathcal{O}_Y)$ is a space with functions equipped with an atlas $(V_i, \psi_i)_{i\in I}$ satisfying conditions (A1), (A2) and (A3), then the isomorphisms $\psi_i^{-1} \circ \varphi_i \colon U_i \to V_i$ induce an isomorphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

Proof. Uniqueness up to canonical isomorphism: Let $(U_i, \varphi_i)_{i\in I}$ and $(V_i, \psi_i)_{i\in I}$ be two atlases modelled on the same gluing data, then for all pair (i,j),

$$\begin{aligned}\psi_j^{-1} \circ \varphi_j\Big|_{U_i\cap U_j} &= \psi_j^{-1} \circ \underbrace{(\varphi_j \circ \varphi_i^{-1})}_{=\varphi_{ji}} \circ \varphi_i\Big|_{U_i\cap U_j} \\ &= \psi_j^{-1} \circ \underbrace{(\psi_j \circ \psi_i^{-1})}_{=\varphi_{ji}} \circ \varphi_i\Big|_{U_i\cap U_j} \\ &= \psi_i^{-1} \circ \varphi_i\Big|_{U_i\cap U_j}\end{aligned}$$

so there is a well-defined map

$$\begin{aligned} f \colon X = \bigcup_{i\in I} U_i &\to \bigcup_{i\in I} V_i = Y \\ (x \in U_i) &\mapsto (\psi_i^{-1} \circ \varphi_i(x) \in V_i)\end{aligned}$$

which induces an isomorphism of spaces with functions.

Existence: Define $\tilde{X} \coloneqq \bigsqcup_{i\in I} X_i$ and let the topology be the final topology with respect to the canonical maps $(X_i \to \tilde{X})_{i\in I}$. Then define $X \coloneqq \tilde{X}/\sim$ where $(i,x) \sim (j,y)$ in $\tilde{X}$ if $x = \varphi_{ij}(y)$. Conditions (1), (2) and (3) show that $\sim$ is reflexive, symmetric and transitive. We equip X with the quotient topology and denote by

$$p \colon \tilde{X} \to X$$

the canonical continuous projection. Let $U_i \coloneqq p(X_i)$. Since $p^{-1}(U_i) = \bigsqcup_{j\in I} X_{ji}$ is open in $\tilde{X}$, U_i is open in X. Moreover, $\bigcup_{i\in I} U_i = X$, so we have an open covering of X. We put $p_i \coloneqq p|_{X_i}$ and we define a sheaf on X by setting

$$\mathcal{O}_X(U) \coloneqq \{f \colon U \to k \mid \forall i \in I, f \circ p_i \in \mathcal{O}_{X_i}(p_i^{-1}(U))\}$$

for all open sets $U \subseteq X$. This defines a sheaf on X, with respect to which $(X, \mathcal{O}_X)$ is a space with functions. Finally, $p_i \colon X_i \to U_i$ is a homeomorphism and, by construction $\mathcal{O}_{U_i} \simeq (p_i)_*\mathcal{O}_{X_i}$ via pullback by p_i. We have thus constructed a space with functions $(X, \mathcal{O}_X)$, equipped with an open covering $(U_i)_{i\in I}$ and local charts

$$\varphi_i \coloneqq p_i^{-1} \colon (U_i, \mathcal{O}_X|_{U_i}) \xrightarrow{\sim} (X_i, \mathcal{O}_{X_i}).$$

It remains to check that $\varphi_i(U_i \cap U_j) = X_{ij}$ and $\varphi_j \circ \varphi_i^{-1} = \varphi_{ji}$ on X_{ij}, but this follows from the construction of $X = \bigsqcup_{i\in I} X_i/\sim$ and the definition of the φ_i's as $p|_{X_i}^{-1}$. □

Example 1.39. Take $k = \mathbb{R}$ or $\mathbb{C}$ equipped with either the Zariski or the usual topology. Consider the spaces with functions $X_1 = k$, $X_2 = k$ and the open sets $X_{12} = k \setminus \{0\} \subseteq X_1$ and $X_{21} = k \setminus \{0\} \subseteq X_2$. Finally, set

$$\begin{aligned} \varphi_{21} \colon X_{12} &\to X_{21} \\ t &\mapsto \frac{1}{t}. \end{aligned}$$

Since this is an isomorphism of spaces with functions, we can glue X_1 and X_2 along $X_{12} \xrightarrow[\varphi_{21}]{\sim} X_{21}$ and define a space with functions $(X, \mathcal{O}_X)$ with an atlas modelled on (X_1, X_2, φ_{21}). We will now identify this space X with the projective line $k\mathbb{P}^1$. By definition, the latter is the set of 1-dimensional vector subspaces (lines) of k^2:

$$k\mathbb{P}^1 := (k^2 \setminus \{0\})/k^\times.$$

Then, we have a covering $U_1 \cup U_2 = k\mathbb{P}^1$, where $U_1 = \{[x_1 : x_2] \mid x_1 \neq 0\}$ and $U_2 = \{[x_1 : x_2] \mid x_2 \neq 0\}$, and we can define charts

$$\begin{aligned} \varphi_1 \colon U_1 &\xrightarrow{\sim} k \\ [x_1 : x_2] &\longmapsto x_2/x_1 \\ [1 : w] &\longleftarrow w \end{aligned}$$

and $\varphi_2 \colon U_2 \to k$ likewise. Then, on the intersection

$$U_1 \cap U_2 = \{[x_1 : x_2] \mid x_1 \neq 0, x_2 \neq 0\}$$

we have a commutative diagram

$$\begin{array}{ccc} U_1 \cap U_2 & & \\ \downarrow{\scriptstyle \varphi_1} & \searrow^{\varphi_2} & \\ X_1 & \xrightarrow{\varphi_{21}} & X_2 \end{array}$$

with $\varphi_i(U_1 \cap U_2)$ open in X_i. In view of the gluing theorem, we can use this to set up a bijection $k\mathbb{P}^1 \to X$ where $X := (X_1 \sqcup X_2)/\sim_{\varphi_{12}}$ and define a topology and a sheaf of regular functions on $k\mathbb{P}^1$ via this identification. Note that this was done without putting a topology on $k\mathbb{P}^1$: the latter is obtained using the bijection $k\mathbb{P}^1 \to X$ constructed above. We now spell out the notion of regular functions thus obtained on $k\mathbb{P}^1$.

Proposition 1.40. *With the identification*

$$k\mathbb{P}^1 = X_1 \sqcup X_2/\sim$$

constructed above, a function $f \colon U \to k$ defined on an open subset $U \subseteq k\mathbb{P}^1$ is an element of $\mathcal{O}_X(U)$ if and only if, for each local chart $\varphi_i \colon U_i \to k$, the function

$$f \circ \varphi_i^{-1} \colon \varphi_i(U_i \cap U) \to k$$

is regular on the open set $\varphi_i(U_i \cap U) \subseteq k$.

Definition 1.41. Let k be a field. An *algebraic k-prevariety* is a space with functions $(X, \mathcal{O}_X)$ such that

(i) X is quasi-compact.

(ii) $(X, \mathcal{O}_X)$ is locally isomorphic to an affine variety.

Remark 1.42. Saying that $(X, \mathcal{O}_X)$ is locally isomorphic to an affine variety means that for $x \in X$, it exists an open neighbourhood $x \in U$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an open subset of an affine variety. Since such an open set is a union of principal open sets, which are themselves affine, one can equivalently ask that $(U, \mathcal{O}_U)$ be affine. Thus:

Proposition 1.43. *A space with functions $(X, \mathcal{O}_X)$ is an algebraic prevariety, if and only if there exists a finite open covering*

$$X = U_1 \cup \ldots \cup U_n$$

such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine variety.

Remark 1.44. As a consequence of the gluing theorem, in order to either construct an algebraic prevariety or put a structure of an algebraic prevariety on a set, it suffices to either define X from certain gluing data $(X_i, X_{ij}, \varphi_{ij})_{(i,j)}$ satisfying appropriate compatibility conditions, or find a covering $(U_i)_{i\in I}$ of a set X and local charts $\varphi_i \colon U_i \to X_i$ such that $X_{ij} = \varphi_i(U_i \cap U_j)$ is open in X_i and $\varphi_j \circ \varphi_i^{-1}$ is an isomorphism of spaces with functions.

In practice, X is sometimes given as a topological space, and $(U_i)_{i\in I}$ is an open covering, with local charts $\varphi_i \colon U_i \to X_i$ that are homeomorphisms. So the condition that X_{ij} be open in X_i is automatic in this case and one just has to check that

$$\varphi_j \circ \varphi_i^{-1} \colon X_{ij} \to X_{ji}$$

induces an isomorphism of spaces with functions. In the present context where X_i and X_j are affine varieties, this means a map

$$X_{ij} \subseteq k^n \to X_{ji} \subseteq k^m$$

between locally closed subsets of k^n and k^m whose components are regular functions.

Example 1.45 (Projective sets)**.** We have already seen that projective spaces $k\mathbb{P}^n$ are algebraic pre-varieties. Let $P \in k[x_0, \ldots, x_n]_d$ be a homogeneous polynomial of degree $d \geq 0$. Although P cannot be evaluated at a point $[x_0 : \ldots : x_n] \in k\mathbb{P}^n$, the condition $P(x_0, \ldots, x_n) = 0$ can be tested, because for $\lambda \in k^x$,

$$P(x_0, \ldots, x_n) = 0 \iff 0 = \lambda^d P(x_0, \ldots, x_n) = P(\lambda x_0, \ldots, \lambda x_n).$$

We use this to define the following *projective sets*

$$\mathcal{V}_{k\mathbb{P}^n}(P_1, \ldots, P_m) = \{[x_0 : \ldots : x_n] \in k\mathbb{P}^n \mid P_i(x_0, \ldots, x_n) = 0 \quad \forall i\}$$

for homogeneous polynomials in $(x_0, \ldots, x_n)$.

We claim that these projective sets are the clsoed sets of a topology on $k\mathbb{P}^n$, called the Zariski topology. A basis for that topology is provided by the principal open sets $D_{k\mathbb{P}^n}(P)$ where P is a homogeneous polynomial. By definition, a regular function on a locally closed subset of $k\mathbb{P}^n$ is locally given by the restriction of a ration fraction of the form

$$\frac{P(x_0, \ldots, x_n)}{Q(x_0, \ldots, x_n)}$$

where P and Q are homogeneous polynomials of the same degree. This defines a sheaf of regular functions on any given locally closed subset X of $k\mathbb{P}^n$.

Proposition 1.46. *A Zariski-closed subset X of $k\mathbb{P}^n$ equipped with its sheaf of regular functions, is an algebraic pre-variety. The same holds for all open subsets $U \subseteq X$.*

Proof. Consider the open covering

$$\begin{aligned} X &= \bigcup_{i=0}^{n} X \cap U_i \\ &= \bigcup_{i=0}^{n} \{[x_0 : \ldots : x_n] \in X \mid x_i \neq 0\}. \end{aligned}$$

Then the restriction to $X \cap U_i$ of the local chart

$$\begin{aligned} \varphi_i \colon U_i &\longrightarrow k^n \\ x = [x_0 : \ldots : x_n] &\longmapsto \underbrace{\left(\frac{x_0}{x_i}, \ldots, \frac{\hat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i} \right)}_{w=(w_0,\ldots,\hat{w}_i,\ldots,w_n)} \end{aligned}$$

sends an x such that $P_1(x) = \ldots = P_m(x) = 0$ to a w such that $Q_1(w) = \ldots = Q_m(w) = 0$ where, for all j,

$$\begin{aligned} Q_j(w) &= P_j(w_0, \ldots, w_{i-1}, 1, w_{i+1}, \ldots, w_n) \\ &= P_j(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \end{aligned}$$

is the dehomogeneisation of P_j. So $\varphi_i(X \cap U_i) = \mathcal{V}_{k^n}(Q_1, \ldots, Q_m) =: X_i$ is an algebraic subset of k^n, in particular an affine variety. It remains to check that $\varphi_i|_{X \cap U_i}$ pulls back regular functions on X_i to regular functions on $X \cap U_i$, and similarly for $(\varphi_i|_{X \cap U_i})^{-1}$. But if f and g are polynomials in $(w_0, \ldots, \hat{w}_i, \ldots, w_n)$,

$$\begin{aligned} \left(\varphi_i^* \frac{f}{g} \right)(x) &= \frac{f(\varphi_i(x))}{g(\varphi_i(x))} \\ &= \frac{f\left(\frac{x_0}{x_i}, \ldots, \frac{\hat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i} \right)}{g\left(\frac{x_0}{x_i}, \ldots, \frac{\hat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i} \right)} \end{aligned}$$

which can be rewritten as a quotient of two homogeneous polynomials of the same degree by multiplying the numerator and denominator by x_i^r with $r \geq \max(\deg(f), \deg(g))$. The computation is similar but easier for $(\varphi_i|_{X \cap U_i})^{-1}$. $\square$

Definition 1.47. A space with functions $(X, \mathcal{O}_X)$ which is isomorphic to a Zariski-closed subset of $k\mathbb{P}^n$ is called a *projective k-variety.*

Lemma 1.48. *The category of affine varieties admits products.*

Proof. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be affine varieties. Choose embeddings $X \subseteq k^n$ and $Y \subseteq k^p$ for some n and p. Then $X \times Y \subseteq k^{n+p}$ is an affine variety, endowed with two morphisms of affine varieties $\mathrm{pr}_1 \colon X \times Y \to X$ and $\mathrm{pr}_2 \colon X \times Y \to Y$. We will prove that the triple $(X \times Y, \mathrm{pr}_1, \mathrm{pr}_2)$ satisfies the universal property of the product of X and Y.

Let $f_X \colon Z \to X$ and $f_Y \colon Z \to Y$ be morphisms of affine varieties. Then define $f = (f_x, f_y) \colon Z \to X \times Y$. This satisfies $\mathrm{pr}_1 \circ f = f_X$ and $\mathrm{pr}_2 \circ f = f_Y$. If we embed Z into some k^m, the components of f_X and f_Y are regular functions from k^m to k^n and k^p. Thus the components of $f = (f_X, f_Y)$ are regular functions $k^m \to k^{n+p}$, i.e. f is a morphism. $\square$

Theorem 1.49. *The category of algebraic pre-varieties admits products.*

Proof. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ algebraic pre-varieties. Let

$$X = \bigcup_{i=1}^{r} X_i \text{ and } Y = \bigcup_{j=1}^{s} Y_j$$

be affine open covers. Then, as a set,

$$X \times Y = \bigcup_{i,j} X_i \times Y_j.$$

By 1.48, each $X_i \times Y_j$ has a well-defined structure of affine variety. Moreover, if $X_i' \subseteq X_i$ and $Y_j' \subseteq Y_j$ are open sets, then $X_i' \times Y_j'$ is open in $X_i \times Y_j$.

So we can use the identity morphism to glue $X_{i_1} \times Y_{j_1}$ to $X_{i_2} \times Y_{j_2}$ along the common open subset $(X_{i_1} \cap X_{i_2}) \times (Y_{j_1} \cap Y_{j_2})$. This defines an algebraic prevariety P whose underlying set is $X \times Y$. Also, the canonical projections $X_i \times Y_j \to X_i$ and $X_i \times X_j \to X_j$ glue together to give morphisms $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$, which coincide with pr_1 and pr_2.

There only remains to prove the universal property. Let $f_x : Z \to X$ and $f_Y : Z \to Y$ be morphisms of algebraic prevarieties and set $f = (f_x, f_y) : Z \to X \times Y$. In particular, $\mathrm{pr}_1 \circ f = f_X$ and $\mathrm{pr}_2 \circ f = f_Y$ as maps between sets. To prove that f is a morphisms of algebraic prevarieties, it suffices to show that this is locally the case. Z is covered by the open subsets $f_X^{-1}(X_i) \cap f_Y^{-1}(Y_j)$, each of which can be covered by affine open subsets $(W_l^{ij})_{1 \le l \le q(i,j)}$. By construction, $f(W_l^{ij}) \subseteq X_i \times Y_j$. So, by the universal property of the affine variety $X_i \times Y_j$, the map $f|_{W_l^{ij}}$ is a morphism of affine varieties. □

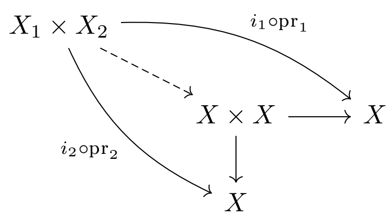
Definition 1.50 (algebraic variety)**.** Let $(X, \mathcal{O}_X)$ be an algebraic pre-variety and $X \times X$ the product in the category of algebraic pre-varieties. If the subset

$$\Delta_X := \{(x, y) \in X \times X \mid x = y\}$$

is closed in $X \times X$, then $(X, \mathcal{O}_X)$ is said to be an *algebraic variety*. A morphism of algebraic varieties $f : X \to Y$ is a morphism of the underlying pre-varieties.

Example 1.51 (of a non-seperated algebraic prevariety)**.** We glue two copies X_1, X_2 of k along the open subsets $k \setminus \{0\}$ using the isomorphism of spaces with functions $t \mapsto t$. The resulting algebraic prevariety is a „line with two origins", denoted by 0_1 and 0_2. For this prevariety X, the diagonal Δ_X is not closed in $X \times X$.

Indeed, if Δ_X were closed in $X \times X$, then its pre-image in $X_1 \times X_2$ under the morphism $f : X_1 \times X_2 \to X \times X$ defined by



where $i_j : X_j \hookrightarrow X$ is the canonical inclusion of X_j into $X = (X_1 \sqcup X_2) / \sim$, would be closed in $X_1 \times X_2$. But

$$\begin{aligned} f^{-1}(\Delta_X) &= \{(x_1, x_2) \in X_1 \times X_2 \mid i_1(x_1) = i_2(x_2)\} \\ &= \{(x_1, x_2) \in X_1 \times X_2 \mid x_j \neq 0 \text{ and } x_1 = x_2 \text{ in } k\} \\ &= \{(x, x) \in k \times k \mid x \neq 0\} \subseteq k \times k = X_1 \times X_2 \end{aligned}$$

which is not closed in $X_1 \times X_2$. In fact, $f^{-1}(\Delta_X) = \Delta_k \setminus \{(0, 0)\} \subseteq k \times k$.

Corollary 1.52. *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be algebraic varieties, then the product in the category of algebraic pre-varieties is an algebraic variety. In particular the category of algebraic varieties admits products.*

Proof. $\Delta_{X \times Y} \simeq \Delta_X \times \Delta_Y \subseteq (X \times X) \times (Y \times Y)$. □

Proposition 1.53. *Affine varieties are algebraic varieties.*

Proof. Let X be an affine variety. We choose an embedding $X \subseteq k^n$. Then $\Delta_X = \Delta_{k^n} \cap (X \times X)$. But

$$\Delta_{k^n} = \{(x_i, y_i)_{1 \le i \le n} \in k^{2n} \mid x_i - y_i = 0\}$$

is closed in k^{2n}. Therefore, Δ_X is closed in $X \times X$ (note that the prevariety topology of $X \times X$ coincides with its induced topology as a subset of k^{2n} by construction of the product prevariety $X \times X$). □

Exercise 1.54. Let $(X, \mathcal{O}_X)$ be an algebraic pre-variety and let $Y \subseteq X$ be a closed subset. For all open subsets $U \subseteq Y$, we set

$$\mathcal{O}_Y(U) := \left\{ h\colon U \to k \mid \forall x \in U \exists x \in \hat{U} \subseteq X \text{ open}, \, g \in \mathcal{O}_X(\hat{U}) \text{ such that } g|_{\hat{U}\cap U} = h|_{\hat{U}\cap U} \right\}.$$

(a) Show that this defines a sheaf of regular functions on Y and that $(Y, \mathcal{O}_Y)$ is an algebraic prevariety.

(b) Show that the canonical inclusion $i_Y : Y \hookrightarrow X$ is a morphism of algebraic prevarieties and that if $f\colon Z \to X$ is a morphism of algebraic prevarieties such that $f(Z) \subseteq Y$, then f induces a morphism $\tilde{f}\colon Y \to Z$ such that $i_Y \circ \tilde{f} = f$.

(c) Show that, if X is an algebraic variety, then Y is also an algebraic variety.

Recall that $k\mathbb{P}^n$ is the projectivisation of the k-vector space k^{n+1}:

$$k\mathbb{P}^n = P(k^{n+1})(k^{n+1} \setminus \{0\})/k^\times.$$

Proposition 1.55 (Segre embedding). *The k-bilinear map*

$$\begin{aligned} k^{n+1} \times k^{m+1} &\longrightarrow k^{n+1} \otimes_k k^{m+1} \simeq k^{(n+1)(m+1)} \\ (x, y) &\longmapsto x \otimes y \end{aligned}$$

induces an isomorphism of algebraic pre-varieties

$$\begin{aligned} P(k^{n+1}) \times P(k^{m+1}) &\xrightarrow{f} \zeta \subseteq P\left(k^{(n+1)(m+1)}\right) = k\mathbb{P}^{nm+n+m} \\ ([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) &\longmapsto [x_0y_0 : \ldots x_0y_m : \ldots : x_ny_0 : \ldots : x_ny_m] \end{aligned}$$

where ζ is a Zariski-closed subset of $k\mathbb{P}^{nm+n+m}$.

Proof. It is clear that f is well-defined. Let us denote by $(z_{ij})_{0\le i\le n, 0\le j\le m}$ the homogeneous coordinates on $k\mathbb{P}^{nm+n+m}$, and call them *Segre coordinates*. Then $f(k\mathbb{P}^n \times k\mathbb{P}^m)$ is contained in the projective variety

$$\begin{aligned} \zeta &= \mathcal{V}\left(\{z_{ij}z_{kl} - z_{kj}z_{il} \mid 0 \le i, k \le n, 0 \le j, l \le m\}\right) \\ &\subseteq P\left(k^{(n+1)(m+1)}\right) \end{aligned}$$

as can be seen by writing

$$f([x], [y]) = \begin{bmatrix} x_0y_0 : & \ldots & : x_0y_m \\ \vdots & & \vdots \\ x_ny_0 : & \ldots & : x_ny_m \end{bmatrix}$$

so that

$$z_{ij}z_{kl} - z_{kj}z_{il} = \begin{vmatrix} x_iy_j & x_iy_l \\ x_ky_j & x_ky_l \end{vmatrix} = 0.$$

The map f is injective because, if $z := f([x], [y]) = f([x'], [y'])$ then there exists (i, j) such that $z \in W_{ij} := \{z \in k\mathbb{P}^{nm+n+m} \mid z_{ij} \neq 0\}$ so $x_iy_j = x'_iy'_j \neq 0$. In particular $\frac{x_i}{x'_i} = \frac{y'_j}{y_j} = \lambda \neq 0$. Since

$$[x_0y_0 : \ldots : x_ny_m] = [x'_0y'_0 : \ldots : x'_ny'_m]$$

means that there exists $\mu \neq 0$ such that, for all (k,l), $x_k y_l = \mu x'_k y'_l$. Taking $k = i$ and $l = j$, we get that $\mu = 1$ and hence, for all k, $x_k y_j = x'_k y'_j$, so $x_k = \frac{y'_j}{y_j} x'_k = \lambda x'_k$. Likewise, for all l, $x_i y_l = x'_i y'_l$, so $y_l = \frac{1}{\lambda} y'_l$. As a consequence $[x_0 : \ldots : x_n] = [x'_0 : \ldots : x'_n]$ and $[y_0 : \ldots : y_m] = [y'_0 : \ldots : y'_m]$, thus proving that f is injective. Note that we have proven that

$$f^{-1}(W_{ij}) = U_i \times V_j$$

where $U_i = \{[x] \in k\mathbb{P}^n \mid x_i \neq 0\}$ and $V_j = \{[y] \in k\mathbb{P}^m \mid y_j \neq 0\}$.

For simplicity, let us assume that $i = j = 0$. The open sets U_0, V_0, W_0 are affine charts, in which f is equivalent to

$$\begin{aligned} k^n \times k^m &\longrightarrow k^{nm+n+m} \\ (u,v) &\longmapsto (v_1, \ldots, v_m, u_1, u_1 v_1, \ldots, u_1 v_m, \ldots, u_n, u_n v_1, \ldots, v_n v_m) \end{aligned}$$

which is clearly regular. In particular $f \mid U_0 \times V_0$ is a morphism of algebraic pre-varieties.

im $f = \zeta$: Let $[z] \in \zeta$. Since the W_{ij} cover $k\mathbb{P}^{nm+n+m}$, we can assume without loss of generality, $z_{00} \neq 0$. Then by definition of ζ, $z_{kl} = \frac{z_{k0} z_{0l}}{z_{00}}$ for all (k,l). If we set

$$([x_0 : \ldots : x_n], [y_0 : \ldots : y_m]) = \left(\left[1 : \frac{z_{10}}{z_{00}} : \ldots : \frac{z_{n0}}{z_{00}} \right], \left[1 : \frac{z_{01}}{z_{00}} : \ldots : \frac{z_{0m}}{z_{00}} \right] \right)$$

we have a well defined point $([x],[y]) \in U_0 \times V_0 \subseteq k\mathbb{P}^n \times k\mathbb{P}^m$, which satisfies $f([x],[y]) = [z]$.

Thus $f^{-1} \colon \zeta \to k\mathbb{P}^n \times k\mathbb{P}^m$ is defined and a morphism of algebraic pre-varieties because, in affine charts $W_0 \xrightarrow{f^{-1}|_{W_0}} U_0 \times V_0$ as above, it is the regular map $(u_{ij})_{(i,j)} \mapsto ((u_{i_0})_i, (u_{0j})_j)$. □

Corollary 1.56. *Projective varieties are algebraic varieties.*

Proof. By 1.54 it suffices to show that $k\mathbb{P}^n$ is an algebraic variety. Let $f \colon k\mathbb{P}^n \times k\mathbb{P}^n \to k\mathbb{P}^{n^2+2n}$ be the Segre embedding. For $[x] \in k\mathbb{P}^n$:

$$f([x],[x]) = \begin{bmatrix} x_0 x_0 : & \ldots & : x_0 x_m \\ \vdots & & \vdots \\ x_n x_0 : & \ldots & : x_n x_m \end{bmatrix}.$$

Thus $f([x],[x])_{ij} = f([x],[x])_{ji}$. Let now $[z] \in \zeta \subseteq k\mathbb{P}^{n^2+2n}$, where ζ is defined in the proof of 1.55, and such that, in Segre coordinates, $z_{ij} = z_{ji}$. Without loss of generality, we can assume $z_{00} = 1$. Set $x_j := z_{0j}$ for $1 \leq j \leq n$. Thus for all (i,j)

$$f([x],[y])_{ij} = x_i x_j = z_{0i} z_{0j} = z_{i0} z_{0j} = z_{ij} z_{00} = z_{ij},$$

i.e.

$$\Delta_{k\mathbb{P}^n} \simeq \{[z] \in \zeta \mid z_{ij} = z_{ji}\}$$

which is a projective and thus closed set of $k\mathbb{P}^n \times k\mathbb{P}^n$. □

1.6 Examples of algebraic varieties

Exercise 1.57. Let $f\colon X \to Y$ be a morphism of algebraic pre-varieties. Assume

(i) Y is a variety.

(ii) There exists an open covering $(Y_i)_{i\in I}$ of Y such that the open subset $f^{-1}(Y_i)$ is a variety.

Show that X is a variety.

Exercise 1.58. Let X be a topological space. Assume that there exists a covering $(X_i)_{i\in I}$ of X by irreducible open subsets such that for all (i,j), $(X_i \cap X_j) \neq \emptyset$. Show that X is irreducible.

1.6.1 Grassmann varieties

Let $0 \le p \le n$ be integers. The Grassmannian $\mathrm{Gr}(p,n)$ is the set of p-dimensional linear subspaces of k^n. In order to endow this set with a structure of algebraic prevariety, there are various possibilities:

(i) To a p-dimensional linear subspace $E \subseteq k^n$, we associate the line $\Lambda^p E \subseteq \Lambda^p k^n \simeq k^{\binom{n}{p}}$, which defines a point in the projective space $k\mathbb{P}^{\binom{n}{p}-1}$.

Claim: The map $\mathrm{Gr}(p,n) \to k\mathbb{P}^{\binom{n}{p}-1}$ is an injective map whose image is a Zariski-closed subset of $k\mathbb{P}^{\binom{n}{p}-1}$.

This identifies $\mathrm{Gr}(n,p)$ canonically to a projective variety. In particular one obtains in this way a structure of *algebraic variety* on $\mathrm{Gr}(p,n)$.

(ii) For the second approach, recall that $\mathrm{GL}(n,k)$ acts transitively on $\mathrm{Gr}(p,n)$. But the identification of k^n to $(k^n)^*$ via the canonical basis of k^n enables one to define, for all $E \in \mathrm{Gr}(p,n)$, a canonical complement $E^\perp \in \mathrm{Gr}(n-p,n)$, i.e. an $(n-p)$-dimensional linear subspace such that $E \oplus E^\perp = k^n$.

So the stabiliser of $E \in \mathrm{Gr}(p,n)$ for the action of $\mathrm{GL}(n,k)$ is conjugate to the subgroup

$$\mathrm{P}(p,n) := \left\{ g \in \mathrm{GL}(n,k) \;\middle|\; \begin{array}{l} g = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \\ \text{with } A \in \mathrm{GL}(p,k), B \in \mathrm{Mat}(p \times (n-p), k), \\ \text{and } C \in \mathrm{GL}(n-p,k) \end{array} \right\}.$$

This shows that the Grassmannian $\mathrm{Gr}(p,n)$ is a homogeneous space under $\mathrm{GL}(n,k)$ and that

$$\mathrm{Gr}(p,n) \simeq \mathrm{GL}(n,k)/\mathrm{P}(p,n)$$

which is useful if one knows that, given an affine algebraic group G and a closed subgroup H, the homogeneous space G/H is an algebraic variety. We will come back to this later on.

(iii) The third uses the gluing theorem. In particular, it also constructs a standard atlas on $\mathrm{Gr}(p,n)$, like the one we had on $k\mathbb{P}^{n-1} = \mathrm{Gr}(1,n)$. The idea is that, in order to determine a p-dimensional subspace of k^n, it suffices to give a basis of that subspace, which is a family of p vectors in k^n. Geometrically, this means that the subspace in question is seen as the graph of a linear map $A\colon k^p \to k^n$.

Take $E \in \mathrm{Gr}(p,n)$ and let $(v_1,\dots,v_p)$ be a basis of E over k. Let M be the $(n\times p)$-matrix representing the coordinates of $(v_1,\dots,v_p)$ in the canonical basis of k^n. Since M has rank p, there exists a $(p\times p)$-submatrix of M with non-zero determinant: We set

$$\begin{aligned} J &:= \{\text{indices } j_1 < \ldots < j_p \text{ of the rows of that submatrix}\} \\ M_J &:= \text{the submatrix in question.} \end{aligned}$$

Note that if $M' \in \mathrm{Mat}(n \times p, k)$ corresponds to a basis $(v_1', \dots, v_p')$, there exists a matrix $g \in \mathrm{GL}(p,k)$ such that $M' = Mg$. But then $(M')_J = (Mg)_J = M_J g$, so

$$\det\ (M')_J = \det\ (M_J g) = \det(M_J)\det(g),$$

which is non-zero if and only if $\det(M_J)$ is non-zero. As a consequence, given a subset $J \subseteq \{1, \dots, n\}$ of cardinal p, there is a well-defined subset

$$G_J := \{E \in \mathrm{G}(p,n) \mid \exists M \in \mathrm{Mat}(n \times p, k), E = \mathrm{im}\ M \text{ and } \det(M_J) \neq 0\}.$$

Moreover, if M satisfies the conditions $E = \mathrm{im}\ M$ and $\det(M_J) \neq 0$, then $(MM_J^{-1})_J = I_p$ and $\mathrm{im}(MM_J^{-1}) = \mathrm{im}\ M = E$. In fact, if $E \in G_J$, there is a unique matrix $N \in \mathrm{Mat}(n \times p, k)$, such that $E = \mathrm{im}\ N$ and $N_J = I_p$, for if N_1, N_2 are two such matrices, the columns of N_2 are linear combinations of those of N_1, thus $\exists g \in \mathrm{GL}(p,k)$ such that $N_2 = N_1 g$. But then

$$I_p = (N_2)_J = (N_1 g)_J = (N_1)_J g = g.$$

So, there is a well-defined map

$$\begin{aligned} \hat{\varphi}_J : G_J &\longrightarrow \mathrm{Hom}(k^J, k^n) \\ E &\longmapsto N \text{ such that } E = \mathrm{im}\ N \text{ and } N_J = I_p \end{aligned}$$

whose image can be identified to the subspace $\mathrm{Hom}(k^J, k^{J^c})$, where J^c is the complement of J in $\{1, \dots, n\}$, via the map $N \mapsto N_{J^c}$. Conversely, a linear map $A \in \mathrm{Hom}(k^J, k^{J^c})$ determines a rank p map $N \in \mathrm{Hom}(k^J, k^n)$ such that $N_J = I_p$ via the formula $N(x) = x + Ax$.

Geometrically, this means that the p-dimensional subspace $\mathrm{im}\ N \subseteq k^n$ is equal to the graph of A. This also means that we can think of G_J as the set

$$\{E \in \mathrm{Gr}(p,n) \mid E \cap k^{J^c} = \{0_{k^n}\}\}.$$

The point is that $\mathrm{im}\ \hat{\varphi}_J = \mathrm{Hom}(k^J, k^{J^c})$ can be canonically identified with the affine space $k^{p(n-p)}$ and that we have a bijection

$$\begin{aligned} \varphi_J \colon G_J &\xrightarrow{\simeq} \mathrm{Hom}(k^J, k^{J^c}) \simeq k^{p(n-p)} \\ E &\longmapsto A \mid \mathrm{gr}(A) = E \\ \mathrm{gr}(A) &\longleftarrow A. \end{aligned}$$

Note that the matrix $N \in \mathrm{Mat}(n \times p, k)$ such that $\mathrm{im}\ N = E$ and $N_J = I_p$ is row-equivalent to $\begin{pmatrix} I_p \\ A \end{pmatrix}$ with $A \in \mathrm{Mat}((n-p) \times p, k)$.

Now, if $E \in G_{J_1} \cap G_{J_2}$, then, for all $M \in \mathrm{Mat}(p \times n, k)$ such that $\mathrm{im}\ M = E$, $\hat{\varphi}_{J_1}(E) = MM_{J_1}^{-1}$ and $\hat{\varphi}_{J_2}(E) = MM_{J_2}^{-1}$. So

$$\begin{aligned} \mathrm{im}\ \hat{\varphi}_{J_1} &= \left\{N \in \mathrm{Hom}(k^{J_1}, k^n) \mid N_{J_1} = I_p, \mathrm{im}\ N_{J_1} = E \text{ and } \det(N_{J_2}) \neq 0\right\} \\ &= \{N \in \mathrm{im}\ \hat{\varphi}_{J_1} \mid \det(N_{J_2}) \neq 0\} \end{aligned}$$

which is open in $\mathrm{im}\ \hat{\varphi}_{J_1} \simeq \mathrm{im}\ \varphi_{J_1}$.

Moreover, for all $N \in \mathrm{im}\ \hat{\varphi}_{J_1}$,

$$\hat{\varphi}_{J_2} \circ \hat{\varphi}_{J_1}^{-1}(N) = NN_{J_2}^{-1}$$

and, by Cramer's formulae, this is a regular function on $\mathrm{im}\ \hat{\varphi}_{J_1}$.

We have therefore constructed a covering

$$\mathrm{Gr}(p,n) = \bigcup_{J \subseteq \{1,\ldots,n\}, \# J = p} G_J$$

of the Grassmannian $\mathrm{Gr}(p,n)$ by subsets G_J that can be identified to the affine variety $k^{p(n-p)}$ via bijective maps $\varphi_J \colon G_j \to k^{p(n-p)}$ such that, for all (J_1, J_2), $\varphi_{J_1}(G_{J_1} \cap G_{J_2})$ is open in $k^{p(n-p)}$ and the map $\varphi_{J_2} \circ \varphi_{J_1}^{-1} \colon \varphi_{J_1}(G_{J_1} \cap G_{J_2}) \to \varphi_{J_2}(G_{J_1} \cap G_{J_2})$ is a morphism of affine varieties. By the gluing theorem, this endows $\mathrm{Gr}(p,n)$ with a structure of algebraic prevariety.

1.6.2 Vector bundles

Definition 1.59. A *vector bundle* is a triple (E, X, π) consisting of two algebraic varieties E and X, and a morphism $\pi \colon E \to X$ such that

(i) for $x \in X$, $\pi^{-1}(\{x\})$ is a k-vector space.

(ii) for $x \in X$, there exists an open neighbourhood U of x and an isomorphism of algebraic varieties

$$\Phi \colon \pi^{-1}(U) \xrightarrow{\simeq} U \times \pi^{-1}(\{x\})$$

such that

(a) $\mathrm{pr}_1 \circ \Phi = \pi|_{\pi^{-1}(U)}$ and

(b) for $y \in U$, $\Phi|_{\pi^{-1}(\{y\})} \colon \pi^{-1}(\{y\}) \to \{y\} \times \pi^{-1}(\{x\})$ is an isomorphism of k-vector spaces.

A morphism of vector bundles is a morphism of algebraic varieties $f \colon E_1 \to E_2$ such that $\pi_2 \circ f = \pi_1$ and f is k-linear in the fibres.

Remark 1.60. In practice, one often proves that a variety E is a vector bundle over X by finding a morphism $\pi \colon E \to X$ and an open covering

$$X = \bigcup_{i \in I} U_i$$

such that $E|_{U_i} := \pi^{-1}(U_i)$ is isomorphic to $U_i \times k^{n_i}$ for some integer n_i, in such a way that, on $U_i \cap U_j$, the morphism

$$\Phi_j \circ \Phi_i^{-1}\Big|_{\Phi_i(\pi^{-1}(U_i \cap U_j))} \colon (U_i \cap U_j) \times k^{n_i} \longrightarrow (U_i \cap U_j) \times k^{n_j}$$

is an isomorphism of algebraic varieties such that the following diagram commutes and $\Phi_j \circ \Phi_i^{-1}$ is linear fibrewise:

$$\begin{array}{ccc} (U_i \cap U_j) \times k^{n_i} & \xrightarrow{\Phi_j \circ \Phi_i^{-1}} & (U_i \cap U_j) \times k^{n_j} \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U_i \cap U_j & \end{array} .$$

In particular $k^{n_i} \simeq k^{n_j}$ as k-vector spaces, so $n_i = n_j$ if $U_i \cap U_j \neq \emptyset$, and $\Phi_j \circ \Phi_i^{-1}$ is necessarily of the form

$$(x, v) \longmapsto (x, g_{ji}(x) \cdot v)$$

for some morphism of algebraic varieties

$$g_{ji} \colon U_i \cap U_j \longrightarrow \mathrm{GL}(n,k).$$

These maps $(g_{ij})_{(i,j)\in I\times I}$ then satisfy for $x \in U_i \cap U_j \cap U_l$

$$g_{lj}(x)g_{ji}(x) = g_{li}(x)$$

and for $x \in U_i$, $g_{ii}(x) = \mathrm{I}_n$.

Proposition 1.61. *If $\pi\colon E \to X$ is a morphism of algebraic varieties and X has an open covering $(U_i)_{i\in I}$ over which E admits local trivialisations*

$$\Phi_i \colon E|_{U_i} = \pi^{-1}(U_i) \xrightarrow{\simeq} U_i \times k^n$$

with $pr_1 \circ \Phi_i = \pi|_{\pi^{-1}(U_i)}$ such that the isomorphisms

$$\Phi_j \circ \Phi_i^{-1} \colon (U_i \cap U_j) \times k^n \longrightarrow (U_i \cap U_j) \times k^n$$

are linear in the fibres, then for all $x \in X$, $\pi^{-1}(\{x\})$ has a well-defined structure of k-vector space and the local trivialisations $(\Phi_i)_{i\in I}$ are linear in the fibres. In particular, E is a vector bundle.

Proof. For $x \in U_i$ and $a, b \in \pi^{-1}(\{x\})$, let

$$a + \lambda b := \Phi_i^{-1}(x, \mathrm{pr}_2(\Phi_i(a)) + \lambda \mathrm{pr}_2(\Phi_i(b))).$$

By using the linearity in the fibres of $\Phi_j \circ \Phi_i^{-1}$, one verifies that this does not depend on the choice of $i \in I$. □

Remark 1.62. Assume given an algebraic prevariety X obtained by gluing affine varieties $(X_i)_{i\in I}$ along isomorphisms $\varphi_{ji} \colon X_{ij} \xrightarrow{\simeq} X_{ji}$ defined on open subsets $X_{ij} \subseteq X_i$, such that $X_{ii} = X_i$, $\varphi_{ii} = \mathrm{Id}_{X_i}$ and $\varphi_{lj} \circ \varphi_{ji} = \varphi_{li}$ on $X_{ij} \cap X_{il} \subseteq X_i$.

Recall that such an X comes equipped with a canonical map $p\colon \bigsqcup_{i\in I} \to X$ such that $p_i := p|_{X_i} \colon X_i \to X$ is an isomorphism onto an affine open subset $U_i := p_i(X_i) \subseteq X$ and, if we set $\varphi_i = p_i^{-1}$, we have $\varphi_j \circ \varphi_i^{-1} = \varphi_{ji}$ on $\varphi_i(U_i \cap U_j)$.

Let us now consider the vector bundle $X_i \times k^n$ on each of the affine varieties X_i and assume that an isomorphism of algebraic prevarieties of the form

$$\begin{aligned} \Phi_{ji} \colon X_{ij} \times k^n &\longrightarrow X_{ji} \times k^n \\ (x,v) &\longmapsto (\varphi_{ji}(x), h_{ji}(x)\cdot v) \end{aligned}$$

has been given, where $h_{ij} \colon X_{ij} \to \mathrm{GL}(n,k)$ is a morphism of algebraic varieties, in such a way that the following compatibility conditions are satisfied:

$$\Phi_{ii} = \mathrm{Id}_{X_{ii}\times k^n}$$

and, for all (i,j,l) and all $(x,v) \in (X_{ij} \cap X_{il}) \times k^n$

$$\Phi_{lj} \circ \Phi_{ji}(x,v) = \Phi_{li}(x,v).$$

Then there is associated to this gluing data an algebraic vector bundle $\pi\colon E \to X$, endowed with local trivialisations $\Phi_i \colon E|_{U_i} \xrightarrow{\simeq} U_i \times k^n$, where as earlier $U_i = p(X_i) \subseteq X$, in such a way that, for all (i,j) and all $(\xi, v) \in (U_i \cap U_j) \times k^n$,

$$\Phi_j \circ \Phi_i^{-1}(\xi, v) = (\xi, g_{ji}(\xi)\cdot v)$$

where $g_{ji}(x) = h_{ji}(\varphi_i(\xi)) \in \mathrm{GL}(n,k)$, so $g_{ii} = \mathrm{I}_n$ on U_i, and, for all (i,j,l) and all $\xi \in U_i \cap U_j \cap U_l$,

$$\begin{aligned} g_{lj}(\xi) g_{ji}(\xi) &= h_{lj}(\varphi_j(\xi)) h_{ji}(\varphi_i(\xi)) \\ &= h_{lj}(\varphi_{ji}(\varphi_i(\xi))) h_{ji}(\varphi_i(\xi)) \\ &= h_{li}(\varphi_i(\xi)) \\ &= g_{li}(\xi). \end{aligned}$$

Indeed, we can simply set

$$E \coloneqq \left(\bigsqcup_{i\in I} X_i \times k^n \right) / \sim$$

where $(x,v) \sim (\varphi_{ji}(x), h_{ji}(x)\cdot v)$, and, by the gluing theorem, this defines an algebraic prevariety, equipped with a morphism $\pi\colon E \to X$ induced by the first projection $\mathrm{pr}_1\colon \bigsqcup_{i\in I}(X_i \times k^n) \to \bigsqcup_{i\in I} X_i$. The canonical map $\hat{p}\colon \bigsqcup_{i\in I}(X_i \times k^n) \to E$ makes the following diagram commute

$$\begin{array}{ccc} \bigsqcup_{i\in I}(X_i \times k^n) & \xrightarrow{\hat{p}} & E \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} \\ \bigsqcup_{i\in I} X_i & \xrightarrow{p} & X \end{array}$$

and it induces an isomorphism of prevarieties

$$\hat{p}|_{X_i\times k^n}\colon X_i \times k^n \xrightarrow{\simeq} E|_{p(X_i)} = \pi^{-1}(p(X_i))$$

such that $\pi \circ \hat{p}|_{X_i\times k^n} = p|_{X_i} \circ \mathrm{pr}_1$. Since $p|_{X_i}$ is an isomorphism between X_i and the open subset $U_i = p(X_i) \subseteq X$ with inverse φ_i, the isomorphism $\hat{p}|_{X_i\times k^n}$ induces a local trivialisation

$$\begin{aligned} \Phi_i\colon E|_{U_i} &\longrightarrow U_i \times k^n \\ w &\longmapsto (\pi(w), v) \end{aligned}$$

where v is defined as above by $\hat{p}(x,v) = w$. Note that $p(x) = \pi(w)$ in this case, and that $\pi^{-1}(\{\pi(w)\}) \simeq k^n$ via $\Phi|_{\pi^{-1}(\{\pi(w)\})}$. As the isomorphism of algebraic prevarieties

$$\Phi_j \circ \Phi_i^{-1}\colon (U_i \cap U_j) \times k^n \longrightarrow (U_i \cap U_j) \times k^n$$

thus defined is clearly linear fibrewise, we have indeed constructed in this way a vector bundle $\pi\colon E \to X$, at least in the category of algebraic prevarieties.

Note that if the prevariety X obtained via the gluing of the X_i is a variety, then we can show that E is actually a variety (because the product variety $U_i \times k^n$ is separated). The rest of the verifications, in particular the fact that for all $(\xi, v) \in U_i \cap U_j \times k^n$

$$\Phi_j \circ \Phi_i^{-1}(\xi, v) = (\xi, h_{ji}(\varphi_i(\xi)) \cdot v)$$

is left to the reader.

Exercise 1.63. Consider the set

$$E \coloneqq \{(\rho, v) \in k\mathbb{P}^1 \times k\mathbb{P}^2 \mid v \in \rho\}$$

and the canonical map $\pi\colon E \to k\mathbb{P}^1$.

Show that E is a vector bundle on $k\mathbb{P}^1$ and compute its „cocycle of transition functions“ g_{10} on the standard atlas (U_0, U_1) of $k\mathbb{P}^1$ with

$$\begin{aligned} \varphi_{10}\colon k \setminus \{0\} &\longrightarrow k \setminus \{0\} \\ t &\longmapsto \frac{1}{t}. \end{aligned}$$