0.1 Examples of algebraic varieties

Exercise 0.1. Let $f\colon X \to Y$ be a morphism of algebraic pre-varieties. Assume

(i) Y is a variety.

(ii) There exists an open covering $(Y_i)_{i\in I}$ of Y such that the open subset $f^{-1}(Y_i)$ is a variety.

Show that X is a variety.

Exercise 0.2. Let X be a topological space. Assume that there exists a covering $(X_i)_{i\in I}$ of X by irreducible open subsets such that for all (i,j), $(X_i \cap X_j) \neq \emptyset$. Show that X is irreducible.

0.1.1 Grassmann varieties

Let $0 \leq p \leq n$ be integers. The Grassmannian $\mathrm{Gr}(p,n)$ is the set of p-dimensional linear subspaces of k^n. In order to endow this set with a structure of algebraic prevariety, there are various possibilities:

(i) To a p-dimensional linear subspace $E \subseteq k^n$, we associate the line $\Lambda^p E \subseteq \Lambda^p k^n \simeq k^{\binom{n}{p}}$, which defines a point in the projective space $k\mathbb{P}^{\binom{n}{p}-1}$.

Claim: The map $\mathrm{Gr}(p,n) \to k\mathbb{P}^{\binom{n}{p}-1}$ is an injective map whose image is a Zariski-closed subset of $k\mathbb{P}^{\binom{n}{p}-1}$.

This identifies $\mathrm{Gr}(n,p)$ canonically to a projective variety. In particular one obtains in this way a structure of *algebraic variety* on $\mathrm{Gr}(p,n)$.

(ii) For the second approach, recall that $\mathrm{GL}(n,k)$ acts transitively on $\mathrm{Gr}(p,n)$. But the identification of k^n to $(k^n)^*$ via the canonical basis of k^n enables one to define, for all $E \in \mathrm{Gr}(p,n)$, a canonical complement $E^\perp \in \mathrm{Gr}(n-p,n)$, i.e. an $(n-p)$-dimensional linear subspace such that $E \oplus E^\perp = k^n$.

So the stabiliser of $E \in \mathrm{Gr}(p,n)$ for the action of $\mathrm{GL}(n,k)$ is conjugate to the subgroup

$$\mathrm{P}(p,n) := \left\{ g \in \mathrm{GL}(n,k) \;\middle|\; \begin{array}{l} g = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \\ \text{with } A \in \mathrm{GL}(p,k), B \in \mathrm{Mat}(p\times(n-p),k), \\ \text{and } C \in \mathrm{GL}(n-p,k) \end{array} \right\}.$$

This shows that the Grassmannian $\mathrm{Gr}(p,n)$ is a homogeneous space under $\mathrm{GL}(n,k)$ and that

$$\mathrm{Gr}(p,n) \simeq \mathrm{GL}(n,k)/\mathrm{P}(p,n)$$

which is useful if one knows that, given an affine algebraic group G and a closed subgroup H, the homogeneous space G/H is an algebraic variety. We will come back to this later on.

(iii) The third uses the gluing theorem. In particular, it also constructs a standard atlas on $\mathrm{Gr}(p,n)$, like the one we had on $k\mathbb{P}^{n-1} = \mathrm{Gr}(1,n)$. The idea is that, in order to determine a p-dimensional subspace of k^n, it suffices to give a basis of that subspace, which is a family of p vectors in k^n. Geometrically, this means that the subspace in question is seen as the graph of a linear map $A\colon k^p \to k^n$.

Take $E \in \mathrm{Gr}(p,n)$ and let $(v_1,\dots,v_p)$ be a basis of E over k. Let M be the $(n\times p)$-matrix representing the coordinates of $(v_1,\dots,v_p)$ in the canonical basis of k^n. Since M has rank p, there exists a $(p\times p)$-submatrix of M with non-zero determinant: We set

$$\begin{aligned} J &:= \{\text{indices } j_1 < \ldots < j_p \text{ of the rows of that submatrix}\} \\ M_J &:= \text{the submatrix in question.} \end{aligned}$$

Note that if $M' \in \mathrm{Mat}(n \times p, k)$ corresponds to a basis $(v_1', \dots, v_p')$, there exists a matrix $g \in \mathrm{GL}(p,k)$ such that $M' = Mg$. But then $(M')_J = (Mg)_J = M_J g$, so

$$\det\,(M')_J = \det\,(M_J g) = \det(M_J)\det(g),$$

which is non-zero if and only if $\det(M_J)$ is non-zero. As a consequence, given a subset $J \subseteq \{1, \dots, n\}$ of cardinal p, there is a well-defined subset

$$G_J := \{E \in \mathrm{G}(p,n) \mid \exists M \in \mathrm{Mat}(n \times p, k), E = \mathrm{im}\; M \text{ and } \det(M_J) \neq 0\}\,.$$

Moreover, if M satisfies the conditions $E = \mathrm{im}\; M$ and $\det(M_J) \neq 0$, then $(MM_J^{-1})_J = I_p$ and $\mathrm{im}(MM_J^{-1}) = \mathrm{im}\; M = E$. In fact, if $E \in G_J$, there is a unique matrix $N \in \mathrm{Mat}(n \times p, k)$, such that $E = \mathrm{im}\; N$ and $N_J = I_p$, for if N_1, N_2 are two such matrices, the columns of N_2 are linear combinations of those of N_1, thus $\exists g \in \mathrm{GL}(p,k)$ such that $N_2 = N_1 g$. But then

$$I_p = (N_2)_J = (N_1 g)_J = (N_1)_J g = g.$$

So, there is a well-defined map

$$\begin{aligned} \hat{\varphi}_J : G_J &\longrightarrow \mathrm{Hom}(k^J, k^n) \\ E &\longmapsto N \text{ such that } E = \mathrm{im}\; N \text{ and } N_J = I_p \end{aligned}$$

whose image can be identified to the subspace $\mathrm{Hom}(k^J, k^{J^c})$, where J^c is the complement of J in $\{1, \dots, n\}$, via the map $N \mapsto N_{J^c}$. Conversely, a linear map $A \in \mathrm{Hom}(k^J, k^{J^c})$ determines a rank p map $N \in \mathrm{Hom}(k^J, k^n)$ such that $N_J = I_p$ via the formula $N(x) = x + Ax$.

Geometrically, this means that the p-dimensional subspace $\mathrm{im}\; N \subseteq k^n$ is equal to the graph of A. This also means that we can think of G_J as the set

$$\{E \in \mathrm{Gr}(p,n) \mid E \cap k^{J^c} = \{0_{k^n}\}\}.$$

The point is that $\mathrm{im}\; \hat{\varphi}_J = \mathrm{Hom}(k^J, k^{J^c})$ can be canonically identified with the affine space $k^{p(n-p)}$ and that we have a bijection

$$\begin{aligned} \varphi_J \colon G_J &\xrightarrow{\simeq} \mathrm{Hom}(k^J, k^{J^c}) \simeq k^{p(n-p)} \\ E &\longmapsto A \mid \mathrm{gr}(A) = E \\ \mathrm{gr}(A) &\longleftarrow A. \end{aligned}$$

Note that the matrix $N \in \mathrm{Mat}(n \times p, k)$ such that $\mathrm{im}\; N = E$ and $N_J = I_p$ is row-equivalent to $\begin{pmatrix} I_p \\ A \end{pmatrix}$ with $A \in \mathrm{Mat}((n-p) \times p, k)$.

Now, if $E \in G_{J_1} \cap G_{J_2}$, then, for all $M \in \mathrm{Mat}(p \times n, k)$ such that $\mathrm{im}\; M = E$, $\hat{\varphi}_{J_1}(E) = MM_{J_1}^{-1}$ and $\hat{\varphi}_{J_2}(E) = MM_{J_2}^{-1}$. So

$$\begin{aligned} \mathrm{im}\; \hat{\varphi}_{J_1} &= \left\{N \in \mathrm{Hom}(k^{J_1}, k^n) \mid N_{J_1} = I_p, \mathrm{im}\; N_{J_1} = E \text{ and } \det(N_{J_2}) \neq 0\right\} \\ &= \{N \in \mathrm{im}\; \hat{\varphi}_{J_1} \mid \det(N_{J_2}) \neq 0\} \end{aligned}$$

which is open in $\mathrm{im}\; \hat{\varphi}_{J_1} \simeq \mathrm{im}\; \varphi_{J_1}$.

Moreover, for all $N \in \mathrm{im}\; \hat{\varphi}_{J_1}$,

$$\hat{\varphi}_{J_2} \circ \hat{\varphi}_{J_1}^{-1}(N) = NN_{J_2}^{-1}$$

and, by Cramer's formulae, this is a regular function on $\mathrm{im}\; \hat{\varphi}_{J_1}$.

We have therefore constructed a covering

$$\mathrm{Gr}(p,n) = \bigcup_{J\subseteq\{1,\ldots,n\},\#J=p} G_J$$

of the Grassmannian $\mathrm{Gr}(p,n)$ by subsets G_J that can be identified to the affine variety $k^{p(n-p)}$ via bijective maps $\varphi_J\colon G_j \to k^{p(n-p)}$ such that, for all (J_1, J_2), $\varphi_{J_1}(G_{J_1}\cap G_{J_2})$ is open in $k^{p(n-p)}$ and the map $\varphi_{J_2}\circ\varphi_{J_1}^{-1}\colon \varphi_{J_1}(G_{J_1}\cap G_{J_2}) \to \varphi_{J_2}(G_{J_1}\cap G_{J_2})$ is a morphism of affine varieties. By the gluing theorem, this endows $\mathrm{Gr}(p,n)$ with a structure of algebraic prevariety.

0.1.2 Vector bundles

Definition 0.3. A *vector bundle* is a triple (E, X, π) consisting of two algebraic varieties E and X, and a morphism $\pi\colon E\to X$ such that

(i) for $x\in X$, $\pi^{-1}(\{x\})$ is a k-vector space.

(ii) for $x \in X$, there exists an open neighbourhood U of x and an isomorphism of algebraic varieties

$$\Phi\colon \pi^{-1}(U) \xrightarrow{\simeq} U\times\pi^{-1}(\{x\})$$

such that

(a) $\mathrm{pr}_1\circ\Phi = \pi|_{\pi^{-1}(U)}$ and

(b) for $y\in U$, $\Phi|_{\pi^{-1}(\{y\})}\colon \pi^{-1}(\{y\}) \to \{y\}\times\pi^{-1}(\{x\})$ is an isomorphism of k-vector spaces.

A morphism of vector bundles is a morphism of algebraic varieties $f\colon E_1\to E_2$ such that $\pi_2\circ f = \pi_1$ and f is k-linear in the fibres.

Remark 0.4. In practice, one often proves that a variety E is a vector bundle over X by finding a morphism $\pi\colon E\to X$ and an open covering

$$X = \bigcup_{i\in I} U_i$$

such that $E|_{U_i} := \pi^{-1}(U_i)$ is isomorphic to $U_i\times k^{n_i}$ for some integer n_i, in such a way that, on $U_i\cap U_j$, the morphism

$$\Phi_j\circ\Phi_i^{-1}\Big|_{\Phi_i(\pi^{-1}(U_i\cap U_j))}\colon (U_i\cap U_j)\times k^{n_i} \longrightarrow (U_i\cap U_j)\times k^{n_j}$$

is an isomorphism of algebraic varieties such that the following diagram commutes and $\Phi_j\circ\Phi_i^{-1}$ is linear fibrewise:

$$\begin{array}{ccc} (U_i\cap U_j)\times k^{n_i} & \xrightarrow{\Phi_j\circ\Phi_i^{-1}} & (U_i\cap U_j)\times k^{n_j} \\ & {\scriptstyle\mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}_1} & \\ & U_i\cap U_j & \end{array} .$$

In particular $k^{n_i}\simeq k^{n_j}$ as k-vector spaces, so $n_i = n_j$ if $U_i\cap U_j\neq\emptyset$, and $\Phi_j\circ\Phi_i^{-1}$ is necessarily of the form

$$(x, v) \longmapsto (x, g_{ji}(x)\cdot v)$$

for some morphism of algebraic varieties

$$g_{ji}\colon U_i \cap U_j \longrightarrow \mathrm{GL}(n,k).$$

These maps $(g_{ij})_{(i,j)\in I\times I}$ then satisfy for $x \in U_i \cap U_j \cap U_l$

$$g_{lj}(x)g_{ji}(x) = g_{li}(x)$$

and for $x \in U_i$, $g_{ii}(x) = \mathrm{I}_n$.

Proposition 0.5. *If $\pi\colon E \to X$ is a morphism of algebraic varieties and X has an open covering $(U_i)_{i\in I}$ over which E admits local trivialisations*

$$\Phi_i\colon E|_{U_i} = \pi^{-1}(U_i) \xrightarrow{\simeq} U_i \times k^n$$

with $pr_1 \circ \Phi_i = \pi|_{\pi^{-1}(U_i)}$ such that the isomorphisms

$$\Phi_j \circ \Phi_i^{-1}\colon (U_i \cap U_j) \times k^n \longrightarrow (U_i \cap U_j) \times k^n$$

are linear in the fibres, then for all $x \in X$, $\pi^{-1}(\{x\})$ has a well-defined structure of k-vector space and the local trivialisations $(\Phi_i)_{i\in I}$ are linear in the fibres. In particular, E is a vector bundle.

Proof. For $x \in U_i$ and $a, b \in \pi^{-1}(\{x\})$, let

$$a + \lambda b := \Phi_i^{-1}(x, \mathrm{pr}_2(\Phi_i(a)) + \lambda \mathrm{pr}_2(\Phi_i(b))).$$

By using the linearity in the fibres of $\Phi_j \circ \Phi_i^{-1}$, one verifies that this does not depend on the choice of $i \in I$. □

Remark 0.6. Assume given an algebraic prevariety X obtained by gluing affine varieties $(X_i)_{i\in I}$ along isomorphisms $\varphi_{ji}\colon X_{ij} \xrightarrow{\simeq} X_{ji}$ defined on open subsets $X_{ij} \subseteq X_i$, such that $X_{ii} = X_i$, $\varphi_{ii} = \mathrm{Id}_{X_i}$ and $\varphi_{lj} \circ \varphi_{ji} = \varphi_{li}$ on $X_{ij} \cap X_{il} \subseteq X_i$.

Recall that such an X comes equipped with a canonical map $p\colon \bigsqcup_{i\in I} \to X$ such that $p_i := p|_{X_i}\colon X_i \to X$ is an isomorphism onto an affine open subset $U_i := p_i(X_i) \subseteq X$ and, if we set $\varphi_i = p_i^{-1}$, we have $\varphi_j \circ \varphi_i^{-1} = \varphi_{ji}$ on $\varphi_i(U_i \cap U_j)$.

Let us now consider the vector bundle $X_i \times k^n$ on each of the affine varieties X_i and assume that an isomorphism of algebraic prevarieties of the form

$$\begin{aligned} \Phi_{ji}\colon X_{ij} \times k^n &\longrightarrow X_{ji} \times k^n \\ (x, v) &\longmapsto (\varphi_{ji}(x), h_{ji}(x) \cdot v) \end{aligned}$$

has been given, where $h_{ij}\colon X_{ij} \to \mathrm{GL}(n,k)$ is a morphism of algebraic varieties, in such a way that the following compatibility conditions are satisfied:

$$\Phi_{ii} = \mathrm{Id}_{X_{ii}\times k^n}$$

and, for all (i,j,l) and all $(x,v) \in (X_{ij} \cap X_{il}) \times k^n$

$$\Phi_{lj} \circ \Phi_{ji}(x,v) = \Phi_{li}(x,v).$$

Then there is associated to this gluing data an algebraic vector bundle $\pi\colon E \to X$, endowed with local trivialisations $\Phi_i\colon E|_{U_i} \xrightarrow{\simeq} U_i \times k^n$, where as earlier $U_i = p(X_i) \subseteq X$, in such a way that, for all (i,j) and all $(\xi, v) \in (U_i \cap U_j) \times k^n$,

$$\Phi_j \circ \Phi_i^{-1}(\xi, v) = (\xi, g_{ji}(\xi) \cdot v)$$

where $g_{ji}(x) = h_{ji}(\varphi_i(\xi)) \in \mathrm{GL}(n,k)$, so $g_{ii} = \mathrm{I}_n$ on U_i, and, for all (i,j,l) and all $\xi \in U_i \cap U_j \cap U_l$,

$$\begin{aligned} g_{lj}(\xi)g_{ji}(\xi) &= h_{lj}(\varphi_j(\xi))h_{ji}(\varphi_i(\xi)) \\ &= h_{lj}(\varphi_{ji}(\varphi_i(\xi)))h_{ji}(\varphi_i(\xi)) \\ &= h_{li}(\varphi_i(\xi)) \\ &= g_{li}(\xi). \end{aligned}$$

Indeed, we can simply set

$$E := \left(\bigsqcup_{i\in I} X_i \times k^n \right) / \sim$$

where $(x,v) \sim (\varphi_{ji}(x), h_{ji}(x)\cdot v)$, and, by the gluing theorem, this defines an algebraic prevariety, equipped with a morphism $\pi\colon E \to X$ induced by the first projection $\mathrm{pr}_1\colon \bigsqcup_{i\in I}(X_i \times k^n) \to \bigsqcup_{i\in I} X_i$. The canonical map $\hat{p}\colon \bigsqcup_{i\in I}(X_i \times k^n) \to E$ makes the following diagram commute

$$\begin{array}{ccc} \bigsqcup_{i\in I}(X_i \times k^n) & \xrightarrow{\hat{p}} & E \\ \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \pi} \\ \bigsqcup_{i\in I} X_i & \xrightarrow{p} & X \end{array}$$

and it induces an isomorphism of prevarieties

$$\hat{p}|_{X_i\times k^n}\colon X_i \times k^n \xrightarrow{\simeq} E|_{p(X_i)} = \pi^{-1}(p(X_i))$$

such that $\pi \circ \hat{p}|_{X_i\times k^n} = p|_{X_i} \circ \mathrm{pr}_1$. Since $p|_{X_i}$ is an isomorphism between X_i and the open subset $U_i = p(X_i) \subseteq X$ with inverse φ_i, the isomorphism $\hat{p}|_{X_i\times k^n}$ induces a local trivialisation

$$\begin{aligned} \Phi_i\colon E|_{U_i} &\longrightarrow U_i \times k^n \\ w &\longmapsto (\pi(w), v) \end{aligned}$$

where v is defined as above by $\hat{p}(x,v) = w$. Note that $p(x) = \pi(w)$ in this case, and that $\pi^{-1}(\{\pi(w)\}) \simeq k^n$ via $\Phi|_{\pi^{-1}(\{\pi(w)\})}$. As the isomorphism of algebraic prevarieties

$$\Phi_j \circ \Phi_i^{-1}\colon (U_i \cap U_j) \times k^n \longrightarrow (U_i \cap U_j) \times k^n$$

thus defined is clearly linear fibrewise, we have indeed constructed in this way a vector bundle $\pi\colon E \to X$, at least in the category of algebraic prevarieties.

Note that if the prevariety X obtained via the gluing of the X_i is a variety, then we can show that E is actually a variety (because the product variety $U_i \times k^n$ is separated). The rest of the verifications, in particular the fact that for all $(\xi, v) \in U_i \cap U_j \times k^n$

$$\Phi_j \circ \Phi_i^{-1}(\xi, v) = (\xi, h_{ji}(\varphi_i(\xi)) \cdot v)$$

is left to the reader.

Exercise 0.7. Consider the set

$$E := \{(\rho, v) \in k\mathbb{P}^1 \times k\mathbb{P}^2 \mid v \in \rho\}$$

and the canonical map $\pi\colon E \to k\mathbb{P}^1$.

Show that E is a vector bundle on $k\mathbb{P}^1$ and compute its „cocycle of transition functions“ g_{10} on the standard atlas (U_0, U_1) of $k\mathbb{P}^1$ with

$$\begin{aligned} \varphi_{10}\colon k \setminus \{0\} &\longrightarrow k \setminus \{0\} \\ t &\longmapsto \frac{1}{t}. \end{aligned}$$