Tut no 14

- Mannigfaltigkeiten, Tangentialraum ~~IIII~~ ✓
- Schwere Integrale | ✓
- e^{At} berechnen ohne AWP. | ✓
- Extrema an NB (Altklausur Albers 2018-1, A1, A3) | ✓
- Extremum klassifizieren ⇝ Was wenn Hessematrix semidefinit? | ✓
- Fixpunkte |

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

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lin DGL $\qquad \dot{x} = ax + b \qquad y' = a(t)y + b(t)$

$$A(t) = \int a(\tau)\,d\tau + c$$

$$y(t) = e^{A(t)}\left[\int_{t_0}^{t} e^{-A(\tau)}\, b(\tau)\,d\tau + y_0\right]$$

$$y' = a(t)\,y \quad \rightsquigarrow \quad y(t) = \underbrace{c}_{c(t)} \cdot e^{A(t)}$$

$$y(t) = \dots + \tilde{c}$$

$$y(t_0) \overset{!}{=} y_0 \quad \dots \quad \Rightarrow \quad \tilde{c} = \_$$

$$\ddot{x} = \alpha\dot{x} + \beta x + \gamma$$

$$\mathrm{Re}(x+iy) = \frac{1}{2}(x+iy + x-iy)$$

$\alpha \qquad \beta = \bar{\alpha}$

$$\mathrm{Re}(\alpha) = \frac{1}{2}(\alpha + \bar{\alpha}) = \frac{1}{2}(\alpha+\beta)$$

$$\mathrm{Im}(\alpha) = \frac{1}{2i}(\alpha - \bar{\alpha}) = \frac{1}{2i}(\alpha-\beta)$$

$$f(x+h) - f(x) = L\,h + o(h)$$

$$D_v f(p) = \lim_{h \to 0} \frac{f(p+hv) - f(p)}{h}$$

$$= \underbrace{Df(p)}_{=:L} \cdot v$$

$\|\cdot\|$ Norm $\Rightarrow$ $d : V \times V \to \mathbb{R}_{\geq 0}$ $\quad (x,y) \longmapsto \|x-y\|$

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Zeige $\nexists$ Norm $\|\cdot\|$ s.d. $\|x-y\| = d(x,y)$

$$2\|x-y\| = \|2(x-y)\|$$

$Q \subseteq \mathbb{R}^2$

$$\int_Q \begin{pmatrix} x \\ y \end{pmatrix} d(x,y)$$

Mannigfaltigkeiten $N \subseteq \mathbb{R}^{n+k}$ n-dim Umgfkt der Klasse C^ℓ

$\overset{\text{def}}{\iff}$ Jeder Punkt $p \in N$ besitzt Umgebung $U \subseteq \mathbb{R}^{n+k}$, s.d. $N \cap U = \underbrace{\pi}_{\text{Permutation der Koordinaten}} (\text{graph } g)$

mit $g : \mathbb{R}^n \xrightarrow{C^\ell} \mathbb{R}^{n+k}$.

Viel wichtiger: $D \subseteq \mathbb{R}^{n+k}$ offen, $f: D \xrightarrow{C^\ell} \mathbb{R}^k$ mit Rang $J_f(p) = k$ $\forall p \in N := f^{-1}(\{0\})$

$\Rightarrow$ $N \subseteq \mathbb{R}^{n+k}$ n-dim Umgfkt.

k Gleichungen

$\Rightarrow$ Von den $n+k$ Dimensionen von $\mathbb{R}^{n+k}$ bleiben $(n+k)-k = n$ übrig

$f^{-1}(\{18\})$ $\quad f = x^5 + x^4 - 7 \overset{!}{=} 18$

$\iff 0 = g = x^5 + x^4 - 7 - 18$

Standardaufgabe für Klausur:

$x^2+y^2+z^2=7$ $\quad x^2+y^2 \; z^2 = |(x,y,z)|_2^2$

$\Rightarrow x^2+y^2+z^2 \le 7 \Rightarrow |(x,y,z)|_2 \le \sqrt{7}$

$M=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2-2z^2=0 \wedge x+y+z=1\}$

Zeige: M C^1-Mannigfaltigkeit. (Welcher Dimension?)

$f:\mathbb{R}^3\to\mathbb{R}^2,\quad (x,y,z)\longmapsto\begin{pmatrix}x^2+y^2-2z^2\\ x+y+z-1\end{pmatrix}\begin{matrix}(\mathrm{I})\\(\mathrm{II})\end{matrix} \Rightarrow M=f^{-1}(\{0\})$

$J_f(x,y,z)=\begin{pmatrix}2x & 2y & -4z\\ 1 & 1 & 1\end{pmatrix}$ ✓

$\operatorname{Rang} J_f(x,y,z)\ge 1$ und $\operatorname{Rang} J_f(x,y,z)=1 \Leftrightarrow \begin{pmatrix}2x\\2y\\-4z\end{pmatrix}=a\begin{pmatrix}1\\1\\1\end{pmatrix}$ für ein $a\in\mathbb{R}$

$\Leftrightarrow 2x=2y=-4z \Leftrightarrow x=y=-2z$

$\Rightarrow x+y+z-1=-2z-2z+z-1=-3z-1$

Ang $(x,y,z)\in M$: $\overset{\mathrm{II}}{\Rightarrow} -3z=1 \Rightarrow z=-\frac{1}{3} \Rightarrow x=y=\frac{2}{3}$

$\overset{\mathrm{I}}{\Rightarrow} \left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2-2\left(\frac{1}{3}\right)^2=0$

$\frac{4}{9}+\frac{4}{9}-\frac{2}{9}=\frac{6}{9}\neq 0$

$\Rightarrow \operatorname{Rang} J_f(x,y,z)=2 \quad \forall (x,y,z)\in M$

Und $f\in C^\infty$.

$\overset{\text{IFT}}{\underset{\text{VL}}{\Rightarrow}}$ M 1-dim. Mgfkt

Sei $(x,y)\in\mathbb{R}^2$. $\quad x+y+z=1 \Rightarrow z=1-x-y$

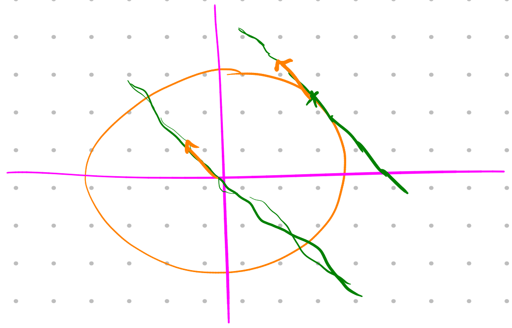
$x^2+y^2-2z^2=0 \Rightarrow x^2+y^2-2(1-x-y)^2=0$

$x^2+y^2-2z^2=0 \Rightarrow 2z^2=x^2+y^2$

$\Rightarrow z=\pm\frac{\sqrt{x^2+y^2}}{2}$

Tangentialraum: (Weniger wichtige) Definition: $M \subseteq \mathbb{R}^{n+k}$ n-dim Mgfkt, $p \in M$

$$T_p M := \{ v \in \mathbb{R}^{n+k} \mid \exists \gamma : (-\varepsilon, \varepsilon) \xrightarrow{C^1} N \text{ mit } \gamma(0) = p,\ \gamma'(0) = v \}$$



Viel wichtiger Situation von oben: $D \subseteq \mathbb{R}^{n+k}$ offen, $f : D \xrightarrow{C^1} \mathbb{R}^k$ $M = f^{-1}(\{0\})$ Mgfkt, $p \in M$

$$T_p M = \ker J_f(p)$$

$M = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2-2z^2=0 \ \wedge\ x+y+z-1=0 \}$

<u>Zurück zur Aufgabe</u> Berechne $T_{(1,1,-1)}M$

Erinnerung: $J_f(x,y,z) = \begin{pmatrix} 2x & 2y & -4z \\ 1 & 1 & 1 \end{pmatrix}$

$(1,1,-1) \in M$, denn: $1+1-1-1=0$ ✓

$1^2+1^2-2(-1)^2=0$ ✓

Setze ein: $J_f(1,1,-1) = \begin{pmatrix} 2 & 2 & +4 \\ 1 & 1 & 1 \end{pmatrix}$

Zu finden: $\ker \begin{pmatrix} 2 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} = \mathrm{Lin}\left[\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right]$

$$\begin{pmatrix} 2 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \cdot(-2) \end{matrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

Lokale Extrema $\quad f: D \xrightarrow{C^1} \mathbb{R}, \quad D \subseteq \mathbb{R}^n$ offen

① $\nabla f(x) = 0$ lösen!

② (Falls $f \in C^2$) $\quad H_f(x,y)$ bestimmen

in Klausur eh immer ...

Falls $H_f(x,y)$ pos. definit $\Rightarrow$ Minimum

neg. definit $\Rightarrow$ Maximum

indefinit $\rightarrow$ Sattelpunkt

Sonst Keine Aussage! $\rightsquigarrow$ "Be extremely clever"

Funktion betrachten zB

① Beschränkt?

② Einsetzen

③ Kompakter Definitions

Altklaus 2018 $\boxed{f: \mathbb{R}^2 \to \mathbb{R}, \ (x,y) \longmapsto \sin(xy).}$ Es ist $f \in C^\infty$.

$$\nabla f(x,y) = \begin{pmatrix} \boxed{y\cos(xy)} \\ x\cos(xy) \end{pmatrix} \qquad \nabla f(x,y) = 0 \Leftrightarrow x = y = 0 \text{ oder } \cos(xy) = 0$$

Es ist $\cos(xy) = 0 \Leftrightarrow xy \in \frac{\pi}{2} + \pi\mathbb{Z}$

$$H_f(x,y) = \begin{pmatrix} -y^2\sin(xy) & \cos(xy) - \sin(xy)xy \\ \cos(xy) - \sin(xy)xy & -x^2\sin(xy) \end{pmatrix}$$ ↙ symmetrisch wegen Schwarz

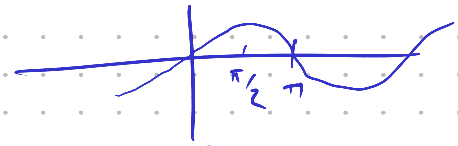
① $(x,y) = (0,0)$:
$$H_f(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

EW von $H_f(0,0)$: $0 = \lambda^2 - 1 \Leftrightarrow \lambda = \pm 1$

$\rightarrow$ indefinit

$\rightarrow$ Sattelpunkt ✓

② $xy \in \frac{\pi}{2} + \pi\mathbb{Z}$.



Falls, $xy \in \frac{\pi}{2} + 2\pi\mathbb{Z}$

$$H_f(x,y) = \begin{pmatrix} -y^2 & -xy \\ -xy & -x^2 \end{pmatrix}$$

negativ semi

Falls $xy \in \frac{3}{2}\pi + 2\pi\mathbb{Z}$

$$H_f(x,y) = \begin{pmatrix} y^2 & xy \\ xy & x^2 \end{pmatrix}$$

positiv semi

$\Rightarrow$ keine Aussage

$|f(x,y)| = |\sin(xy)| \leq 1$ und $\sin(xy) = 1$ für $xy \in \frac{\pi}{2} + 2\pi\mathbb{Z}$

$\sin(xy) = -1$ für $xy \in \frac{3}{2}\pi + 2\pi\mathbb{Z}$

$\Rightarrow$ (x,y) Extrema

Blatt 6 A1

$$H_f(\alpha, -\alpha) = \begin{pmatrix} 6\alpha & 3\alpha \\ 3\alpha & 6\alpha \end{pmatrix}$$

$$\det H_f(\alpha, -\alpha) = (6\alpha)^2 - (3\alpha)^2$$
$$= 36\alpha^2 - 9\alpha^2 > 0$$

Extrema unter NB

$D \subseteq \mathbb{R}^{n+k}$, $f: D \xrightarrow{C^1} \mathbb{R}$ $\qquad g: D \xrightarrow{C^1} \mathbb{R}^{k}$

zu extremieren $\qquad$ Nebenbedingung

f lokales Extremum in $p \in \{g=0\}$ unter NB $g=0$ und $\nabla g_1, \dots, \nabla g_k$ l.u.

Lagrange $\Rightarrow$ $\exists \lambda_i \in \mathbb{R}$, $\nabla f(p) = \lambda_1 \nabla g(p) + \dots + \lambda_k \nabla g(p)$

Multiplikatoren

Klausur 2020

$f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto (x-4)^2 + (y+2)^2$

NB $g: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x^2 + y^2 - 5 = 0$

Finde mögl. lokale Extrema <u>und</u> entscheide ob tatsächlich eine Lösung

$\nabla f(x,y) = \begin{pmatrix} 2x-8 \\ 2y+4 \end{pmatrix}$ $\qquad \nabla g(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2x = 0 \\ 2y = 0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases}$

Sei $p=(x,y)$ lokal. Extremum unter NB $g=0$. Da $\nabla g(x,y) = 0 \Leftrightarrow x=y=0 \rightarrow g(x,y) \neq 0$ und $f, g \in C^1$.

Lagrange $\Rightarrow$ $\exists \lambda \in \mathbb{R}$ sd $\qquad \nabla f(x,y) = \lambda \nabla g(x,y)$

$$\Rightarrow \begin{pmatrix} 2x-8 \\ 2y+4 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix} \Leftrightarrow \begin{cases} 2x - 8 = \lambda 2x \\ 2y + 4 = \lambda 2y \end{cases}$$

$2x - 8 = \lambda 2x$ $\quad \leftarrow$ da $x=0 \Rightarrow -8=0$ ↯

$x=0 \Rightarrow -8 = 0$ $\qquad$ y analog

$\boxed{x \neq 0 \Leftrightarrow y \neq 0}$ $\quad \begin{cases} 2xy - 8y = \lambda 2xy \\ 2xy + 4x = \lambda 2xy \end{cases}$

$\Leftrightarrow \cancel{2xy} - 8y = \cancel{2xy} + 4x$

$\Leftrightarrow -2y = x$

$NB: 0 = g(x,y) = x^2 + y^2 - 5 \quad \Leftrightarrow (-2y)^2 + y^2 - 5 = 0$

$\Leftrightarrow 5y^2 = 5$

$\Leftrightarrow y_1 = 1, \quad y_2 = -1$

$\Leftrightarrow x_1 = -2 \quad x_2 = 2$

Mögl. lokale Extrema $(2,-1)$, $(-2,1)$

$2x_1 - 8 = 2\lambda_{1,1} x_1 \Rightarrow 4-8 = 4\lambda_1 \Rightarrow \lambda_{1,1} = -1$

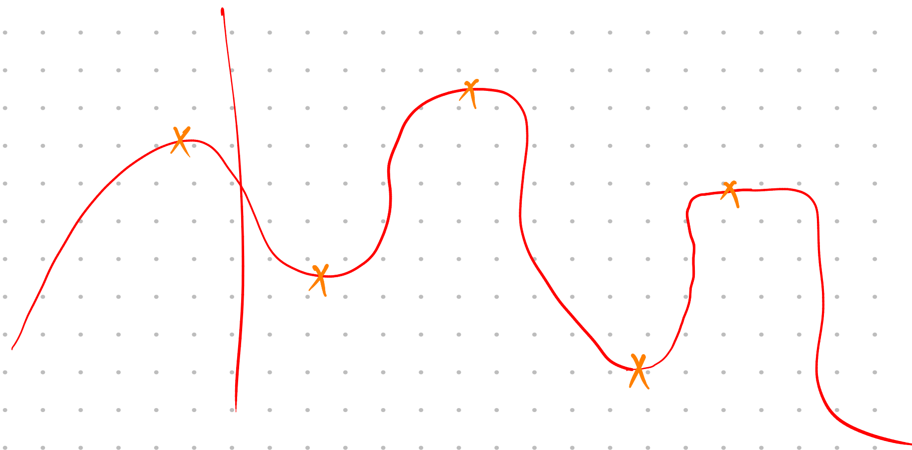
$x_2 \qquad \lambda_{1,2} x_2 \Rightarrow \qquad \lambda_{1,2} = -$

$\{g=0\} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 5\}$ $\quad g$ stetig $\Rightarrow$ $\{g=0\}$ kompakt

$= \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\|_2^2 = 5\}$

$= g^{-1}(\{0\})$

f stetig $\Rightarrow$ Ana 1 $\quad \exists\, x_{min}, x_{max} \in \{g=0\}$ und nur $(2,-1), (-2,1)$ erfüllen notw. Bed $\Rightarrow$ beides sind Extrema unter NB



Fixpunkte (M,d) metrischer Raum, $f: M \to M$.

$x \in M$ heißt Fixpunkt $\Leftrightarrow$

Banachscher Fixpunktsatz: Falls f Lipschitzstetig mit Lipschitzkonstante $L < 1$ und (M,d) vollständig $\Rightarrow$ $\exists!$ Fixpunkt.

Albers 2016 Finde 2 Beispiele für vollst. metr. Raum und Lipschitz-stetige Abb $f: M \to M$ mit zwei verschiedenen Fixpunkten.

Schwere Integrale lösen

Tricks: partielle Integration ✓, Substitution ✓, "Stammfkt sehen", Partialbruchzerlegung ✓

$u := x^2 \quad \frac{du}{dx} = 2x$

$u = x^2 - 1, \; \frac{du}{dx} = 2x$

Löse $\int_0^{\pi/2} x\sin(x)\,dx$, $\int_0^{10} x e^{-x^2}\,dx$, $\int_0^1 \frac{x}{\boxed{x^2+1}}\,dx$, $\int_2^3 \frac{1}{x^2-1}\,dx$

$\int_0^{\pi} \sin(x)\, e^{\boxed{\cos(x)}}\,dx$, $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, $\int_0^{\pi/2} \sin^2(x)\,dx$, $\int_0^1 \frac{1}{x^2+x-6}\,dx$ ✓

$u := \cos(x) \quad \frac{du}{dx} = -\sin(x)$

$$\int_0^{\pi/2} \underbrace{x}_{g}\, \underbrace{\sin(x)}_{f'}\,dx = -\cos(x)\,x \Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos(x))\,dx = 0 + \sin(\pi/2) - \sin(0)$$

$$= 1$$

$$\int_a^b g f' = g f \Big|_a^b - \int_a^b g' f$$

$$\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3 \frac{1}{(x+1)(x-1)}\,dx = \frac{1}{2}\int_2^3 \frac{1}{x-1}\,dx - \frac{1}{2}\int_2^3 \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\left[\ln(x-1)\Big|_2^3 - \ln(x+1)\Big|_2^3\right]$$

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$1 = A(x+1) + B(x-1)$ $\quad x=1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

$x=-1 \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$

$$\frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1} = \frac{1}{2}\left[\frac{x+1-(x-1)}{(x-1)(x+1)}\right] = \frac{\cancel{1}}{\cancel{2}}\,\frac{\cancel{2}}{(x+1)(x-1)}$$

$$\int_0^{\pi} \sin(x)\, e^{\cos(x)}\, dx = \int_{u(0)=1}^{u(\pi)=-1} e^u \frac{du}{-\sin(x)} \sin(x) = \int_{-1}^{1} e^u\, du = e - \frac{1}{e}$$

Substitution

$u(x) := \cos(x) \in C^1$

$$\frac{du}{dx} = -\sin(x) \quad " \Rightarrow \quad du = -\sin(x)\, dx "$$

$$\Rightarrow \quad dx = \frac{du}{-\sin(x)}$$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \int_{v(0)}^{v(1)} \frac{1}{\sqrt{1-\sin^2(v)}} \cos(v)\, dv = \int_0^{\pi/2} \frac{1}{\sqrt{\cos^2(v)}} \cos(v)\, dv = \int_0^{\pi/2} dv = \frac{\pi}{2}$$

$\rightsquigarrow \quad x = \sin(v)$

$v(x) := \arcsin(x)$

$$" \frac{dv}{dx} = \frac{1}{\frac{dx}{dv}} = \frac{1}{\cos(v)} \quad \Rightarrow \quad dv = \frac{1}{\cos(v)}\, dx \quad \Rightarrow \quad dx = \cos(v)\, dv "$$

$$\int_0^1 \frac{1}{x^2+x-6}\, dx = \frac{1}{5}\int_0^1 \frac{1}{x-2}\, dx - \frac{1}{5}\int_0^1 \frac{1}{x+3}\, dx$$

$$0 = x^2 + x - 6 = (x-2)(x+3)$$

$-2 \cdot 3 \rightsquigarrow -2+3 = 1$

$= -3 \cdot 2 \rightsquigarrow -3+2 = -1$

$= -1 \cdot 6$

$= -6 \cdot 1$

$$\frac{1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$\Rightarrow \quad 1 = A(x+3) + B(x-2)$

$x=2 \Rightarrow \quad 1 = 5A \Rightarrow A = \frac{1}{5}$

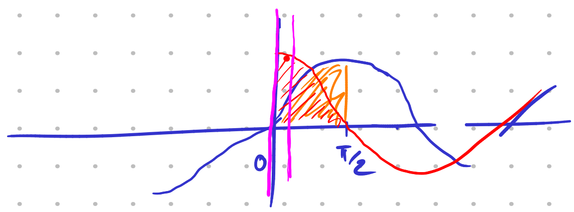
$x=-3 \Rightarrow \quad 1 = -5B \Rightarrow B = -\frac{1}{5}$

$$\frac{1}{5}\,\frac{1}{x-2} - \frac{1}{5}\,\frac{1}{x+3} = \frac{1}{5}\left\{ \frac{x+3-(x-2)}{(x-2)(x+3)} \right\} = \cancel{\frac{1}{5}}\left(\frac{\cancel{5}}{(x-2)(x+3)} \right) \checkmark$$

Satz von Vieta $\quad (x-\alpha)(x-\beta) = x^2 - \alpha x - \beta x + \alpha\beta$

$$= x^2 - (\alpha+\beta)x + \alpha\beta$$

$$\int_0^{\pi/2} \sin^2(x)\, dx = \int_0^{\pi/2} \cos^2(x)\, dx$$



$$2\int_0^{\pi/2} \sin^2(x)\,dx = \int_0^{\pi/2} \sin^2(x)\,dx + \int_0^{\pi/2} \sin^2(x)\,dx$$

$$= \int_0^{\pi/2} \left[\sin^2(x) + \cos^2(x)\right] dx$$

$$= \int_0^{\pi/2} 1\,dx = \pi/2$$

$$\Rightarrow \int_0^{\pi/2} \sin^2(x)\,dx = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$$